ON THE

UNDULATORY THEORY OF OPTICS,

Designed for the Use of Students in the University,

BY

GEORGE BIDDELL AIRY, M.A.

ASTRONOMER ROYAL,
FORMERLY FELLOW OF TRINITY COLLEGE, AND LATE PLUMIAN PROFESSOR OF
ASTRONOMY AND EXPERIMENTAL PHILOSOPHY IN THE
UNIVERSITY OF CAMBRIDGE.

A NEW EDITION.

London and Cambridge:
MACMILLAN AND CO.

Cambridge:
PRINTED BY C. J. CLAY, M.A.
AT THE UNIVERSITY PRESS.

PREFACE.

THIS Treatise on the Undulatory Theory of Optics was first printed in the year 1831, as the last Essay in the Second Edition of a series of "Mathematical Tracts;" and was subsequently reprinted, occupying a similar place, in successive editions of that work.

At the suggestion of the Publisher, it has now, with my approval, been printed in a separate form. Its arrangement and details are however the same, without material alteration, as in the Fourth Edition of the "Mathematical Tracts."

I am happy to state that the work has been passed through the press under the superintendence of Robert Morton, Esq., B.A. of St Peter's College. Every security is thus given for the general accuracy of the publication.

G. B. AIRY.

ROYAL OBSERVATORY, GREENWICH,

EXTRACT FROM THE PREFACE TO THE SECOND EDITION OF THE MATHEMATICAL TRACTS.

"THE Undulatory Theory of Optics is presented to the reader as having the same claims to his attention as the Theory of Gravitation: namely, that it is certainly true, and that, by mathematical operations of general elegance, it leads to results of great interest. With regard to the evidence for this theory; if the simplicity of a hypothesis, which explains with accuracy a vast variety of phænomena of the most complicated kind, can be considered a proof of its correctness, I believe there is no physical theory so firmly established as the theory in question. This can be felt completely, perhaps, only by the person who has both observed the phænomena and made the calculations; as to my own pretensions to the former qualification, I shall merely state that I have repeated nearly every experiment alluded to in the following Tract. This character of certainty I conceive to belong only to what may be called the *geometrical* part of the theory: the hypothesis, namely, that light consists of undulations depending on transversal vibrations, and that these travel with certain velocities in different media according to the laws here explained. The *mechanical* part of the theory, as the suppositions relative

to the constitution of the ether, the computation of the intensity of reflected and refracted rays, &c., though generally probable, I conceive to be far from certain.

"The plan of this Tract has therefore been to include those phænomena only which admit of calculation. Many subjects are thus excluded (for instance, the absorption of light by coloured media) for which supplementary theories are still wanting. On the other hand, the investigations are applied only to phænomena which actually have been observed: as I have thought it useless to suppose imaginary combinations, where the real conditions of experiment offer so great variety.

"The second investigation of the intensity of light reflected from a glass surface, and that of the nature of light reflected internally and totally from glass, were written as a conjectural restoration of Fresnel's investigations, when his paper was supposed to be lost. That paper has since been found and published: the only alteration which it appeared necessary to make is contained in the note attached to the latter."

ON THE

UNDULATORY THEORY OF OPTICS.

ON UNDULATIONS GENERALLY.

PROP. 1. To explain the nature of an Undulation.

1. The characteristic of an undulation is, the continued transmission in one direction of a *relative state* of particles amongst each other, while the motion of each particle separately considered is a reciprocating motion. The disturbance of the particles from their state of rest, and their motion, may be in any direction whatever.

2. For example: in fig. 1, let the line (α) represent a number of particles in their position of rest: and suppose that in consequence of a disturbance they are at a given time T in the position (β): at the time $T+\frac{\tau}{4}$, in the position (γ): at the time $T+\frac{2\tau}{4}$, in the position (δ): at the time $T+\frac{3\tau}{4}$, in the position (ϵ): and at the time $T+\tau$, in the position (ζ): and in intermediate positions at times intermediate to these. At the time T the particles are in the state of greatest condensation about a, a', and a''. Suppose we fix our attention on one of these condensed groups, as for instance that of which a' is the center. At the time $T+\frac{\tau}{4}$ the center of the condensed group has glided from a' to d', not by the motion of all the particles in that direction, but by such a difference of motions that the particles about a' are not so close together

as they were, and the particles about d' are closer together than they were. At the time $T+\frac{2\tau}{4}$ the point of greatest condensation has advanced to g', precisely the point where at the time T there was the least condensation: at the time $T+\frac{3\tau}{4}$, it has advanced to k': and at the time $T+\tau$, to a''. The particles are now, it may be observed, in just the same state as at the time T, for a'' was then the center of a condensed group. After this, everything goes on in the same manner, beginning at the time $T+\tau$, as it did beginning at the time T. All that we have said with respect to the condensed mass about a' applies to those about a, a'', and a'''. Now if these motions were really going on before our eyes, we should see several *condensations* (not the condensed particles) passing uniformly and continuously from the left to the right of the line of particles.

But if we fix our attention on any one of these particles, we shall see that it has a reciprocating or oscillating motion. The particle a is advancing from T to $T+\frac{\tau}{4}$, when it has attained its greatest advance: it recedes then to $T+\frac{3\tau}{4}$: it then advances again. The particle d advances from T (when it is at its minimum advance) to $T+\frac{2\tau}{4}$: it then recedes to $T+\tau$. The particle g recedes to $T+\frac{\tau}{4}$, then advances to $T+\frac{3\tau}{4}$, then recedes. And so for the others. The varying state of particles which we have here supposed, satisfies therefore the conditions mentioned in (1), and therefore this is an instance of undulation, the motion of every particle being backwards and forwards in the same line as the direction of transmission of the wave*.

* This is the kind of undulation which in the air produces sound, and is the only kind which, till within a few years, was used for the explanation of the phenomena of Optics.

3. As another example, let (β), (γ), (δ), (ϵ), (ζ), of fig. 2, represent successive states of the particles which when at rest were in the position (α). If we fix our attention on one of the most elevated parts, as for instance k, at T, we find that at $T+\frac{\tau}{4}$ the elevation has passed to a'; at $T+\frac{2\tau}{4}$ to d': &c.: though the particles have had no motion whatever in that direction. And if these motions were actually before us, we should see several elevations passing uniformly and continuously from the left to the right. But if we fixed our attention on any one particle, we should see that it has an oscillating motion above and below the line. The particle a for instance, is at its greatest elevation at $T+\frac{\tau}{4}$, and at its greatest depression at $T+\frac{3\tau}{4}$: d is at its greatest depression at T, at its greatest elevation at $T+\frac{2\tau}{4}$, and at its greatest depression at $T+\tau$: and so for the others. This varying state of particles is therefore another instance of undulation, the motion of every particle being at right angles to the direction of transmission of the wave.

We might conceive more complicated cases of undulation, as when the motion of the particles is compounded of the two motions supposed in these two cases*; or when there is one motion similar to that represented in fig. 2 in the plane of the paper, and another perpendicular to that plane†; &c. The last of these suppositions is that to which we shall hereafter refer the phænomena of polarization, and of Optics in general.

PROP. 2. The length of a wave does not depend on the extent of vibration of each particle.

4. It is easily seen that the interval between corresponding points of two waves of condensation in fig. 1 (which is

* This is the kind of undulation which takes place on the surface of deep water in a calm.

† This is the undulation of a musical string.

the distance from a to a', a' to a'', &c. at T, or the distance from d to d', d' to d'', &c. at $T+\frac{\tau}{4}$, &c.) is wholly independent of the extent of vibration of each particle. For if each particle vibrated only half as far as is now supposed, still at T, a would be a point where the particles are most condensed, and a' would be the next point where they are most condensed, and a'' the next, &c. The interval between similar points of two waves (which we shall call the *length of a wave*, and shall always denote by the letter λ) would be the same as at present: the only difference would be that the particles about a, a', &c. would not be so closely condensed, nor those about g, g', &c. so widely separated as at present. Similarly the length of a wave in fig. 2 would be unaltered if the vibration of the particles were altered in any ratio: the only difference would be that the elevation of the high points and the depression of the low points would be altered in that ratio.

PROP. 3. The length of a wave depends on the velocity of transmission, and on the time of vibration of each particle.

5. In the cases both of fig. 1 and of fig. 2 (and in every other conceivable case of a continued sequence of waves) we see that every particle has returned to the same state at $T+\tau$ as at T, that is, that the vibration of every particle is completed in the time τ. But in this time the wave has appeared to glide over a space equal to the interval between corresponding points of two waves, or λ. Hence we find,

Space described by the wave in the time of vibration of a particle $=\lambda$.

$$\text{Velocity of wave} = \frac{\lambda}{\text{time of vibration of a particle}}.$$

PROP. 4. To express algebraically the transmission of an undulation.

6. The quantity for which we shall seek an expression is, the distance of any point from its point of rest, in a function of the time and of the distance of that point of rest from some fixed point. Let x be the original distance of any point in

e line (a) from some fixed point: to find an expression for disturbance at the time t, in a function of x and t, con-tent with the conditions of an undulation. By the original scription of an undulation (1), putting v for the velocity of e wave's transmission, it is easily seen that whatever be the ite of disturbance at the time t of a particle whose original linate is x, the same state of disturbance must hold at the ne $t+t'$ for a particle whose original ordinate is $x+vt'$. Or ϕ express the form of the function,

$$\phi(x, t) \text{ must } = \phi\{(x+vt'), (t+t')\},$$

iatever be the value of t'. It will be found on trial that $vt-x$) satisfies this condition, ϕ being any function whatever. For putting $t+t'$ for t, and $x+vt'$ for x, it becomes

$$\phi\{v(t+t')-(x+vt')\}=\phi(vt-x),$$

e same as before. But it may be found analytically thus. xpanding the second side we have

$$\phi(x, t)=\phi(x, t)+\frac{d.\phi(x, t)}{dx}vt'+\frac{d.\phi(x, t)}{dt}t'+\&c.,$$

$$\text{or } v\frac{d.\phi(x, t)}{dx}+\frac{d.\phi(x, t)}{dt}=0,$$

e general solution of which gives

$$\phi(x, t)=\phi(vt-x)^*.$$

7. This expression however is too general to be of much e to us, and we will choose a particular form that will be re convenient. Suppose we fix on this condition to deter-ne the form of the function: the vibration of each particle ill follow the same law as the vibration of a cycloidal pen-

* This is the expression found, by investigation from *mechanical* principles, the disturbance of the particles of air when sound passes along a tube of orm bore, or for the disturbance of an elastic string (as that of a musical rument) fixed at both ends.

dulum. The distance of a cycloidal pendulum from its place of rest is expressed by

$$a \sin\left\{t\sqrt{\left(\frac{g}{l}\right)} + C\right\}, \text{ or } a \sin(nt + C).$$

The required function then for the disturbance of a particle is

$$a \sin\left\{\frac{n}{v}(vt - x) + A\right\}.$$

For while we consider the motion of a single particle only, x is constant, and the expression is

$$a \sin(nt + C), \text{ where } C = A - \frac{nx}{v}.$$

At the same time it is a function of $vt - x$, and therefore satisfies the condition requisite for an undulation. We will therefore assume as the expression for the disturbance

$$a \sin\left(nt - \frac{nx}{v} + A\right).$$

But it is plain that, without any loss of generality, we may get rid of A by altering the origin of time from which t is reckoned, or the origin of linear measure from which x is reckoned. We may therefore take

$$a \sin\left(nt - \frac{nx}{v}\right)$$

as the expression for the disturbance when one undulation only (consisting of an indefinite number of similar waves) is considered.

8. A form somewhat more convenient may be given thus. The expression

$$a \sin\left(nt - \frac{nx}{v} + A\right)$$

goes through all its periodical values while nt increases by

2π, or while t increases by $\frac{2\pi}{n}$. $\frac{2\pi}{n}$ is therefore the time of vibration of a particle. But by (5)

$$v = \frac{\lambda}{\text{time of vibration}} = \frac{\lambda}{\frac{2\pi}{n}} = \frac{\lambda n}{2\pi}.$$

Consequently $n = \frac{2\pi v}{\lambda}$, and therefore the disturbance

$$= a \sin\left\{\frac{2\pi}{\lambda}(vt - x) + A\right\}^*,$$

for which we may put

$$a \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

when a single undulation only is considered. It is to be observed that a is the maximum vibration of any particle.

Prop. 5. To explain the interference of undulations.

9. By *interference* is meant the co-existence of two undulations in which the length of a wave is the same. The conception of interference is not in any circumstances easy†, and it is more particularly difficult with regard to Physical Optics, from our ignorance of the physical causes to which the undulation is due.

* This is the form of the function tacitly assumed by Newton for the disturbance of particles of air, in his investigation of the velocity of sound. (*Principia*, Lib. II. Prop. 47).

† The simplest illustration is perhaps to be found in the crossing of two waves on the surface of water, each of which affects the surface in the same manner as if the other were not there. If we conceive two series of waves, produced by agitating the surface at two points, to spread in circular forms with equal and uniform velocities, and if one agitation was created a little before the other, so that the wave proceeding from one has proceeded as far (in a given direction) as the hollow between two waves proceeding from the other, then it may be imagined that at every point where this holds, the elevation of one wave may exactly fill up the hollow of the other, and the surface will be, in fact, undisturbed.

10. *If we investigate, from the known properties of air, the motion of the particles (supposed parallel to a fixed line), we find this differential equation for the disturbance of a particle

$$\frac{d^2X}{dt^2} - v^2\frac{d^2X}{dx^2} = 0$$

(v^2 being a constant, $= mgH$ in the common notation, and X being the disturbance from the state of rest), and the solution is

$$X = \phi(vt - x) + \psi(vt + x),$$

where the form of the functions is to be determined by the initial circumstances. Or if we suppose the wave of air to move only in one direction, the expression for the disturbance will be $\phi(vt - x)$. And this may be divided into several different expressions,

$$\psi(vt - x) + \chi(vt - x) + v(vt - x) + \&c.$$

where the form of each is to be determined by the initial circumstances, or by the cause of the undulation. If there was only a single original cause for the undulations, there would be only a single term $\psi(vt - x)$ to be preserved. But if there were several distinct original causes for the undulations, there would be a single term corresponding to each of these to be preserved, and the whole disturbance would be the sum of all these terms. And it is particularly to be remarked, that the whole disturbance thus found as the effect of all the original causes together, is precisely the sum (with their proper signs) of a number of disturbances, each of which would have been produced by one of the original causes acting separately.

11. Now if we examine to what this property of the solution of the differential equation (namely that it can be broken up into several parts all similar to each other and to the whole) is due, we find it is owing to the circumstance that the differential coefficients of u were raised only to the first power in

* The reader who is not familiar with the investigation of the problem of Sound may omit the next three articles.

the equation, or (to express it in other words) that the equation was linear. For the differential coefficient of the sum of a number of functions is the same as the sum of the differential coefficients: but the square of the differential coefficient of a sum of functions is not the same as the sum of their squares, &c. If then the differential coefficients (and the unknown quantity itself if it enters in the equation) be all of the first dimension, the substitution of a sum of functions is the same as the sum of their substitutions separately, and therefore if each of those functions satisfies the equation, their sum will satisfy the equation. But if they are raised to a higher power, the substitution of the sum is not the same as the sum of the substitutions, and therefore if each function satisfies the equation, their sum will not.

12. If now we retrace the steps of the investigation for air, it will be seen that the *linearity* of the differential equation depends upon this physical fact, that upon altering by a small quantity the relative position of particles, the forces which they exert undergo variations very nearly proportional to that small quantity. And in any other case where this holds, the equations will be linear; and the wave-disturbance of any particle, produced by a number of agitating causes, will be the sum of all the wave-disturbances which these causes would singly have produced. We can hardly conceive any law of constitution of a medium in which undulations are propagated, where this does not hold, and we shall therefore suppose it to be true for light.

13. Taking it then as a fact that the disturbance of every particle produced by two co-existent undulations will be the sum of the disturbances which they would produce separately, we will consider the nature of the disturbance produced by the superposition of two such undulations as those treated of in (7) and (8), each of which is represented geometrically by fig. 1, if the vibrations are in the direction of the wave's transmission, and by fig. 2, if they are perpendicular to that direction. For convenience of figure, we will suppose them of the latter class: but all that we say will apply as well to the former. We will suppose the length of a wave the same in both undulations. In fig. 3, let the Italic letters of (a)

represent the state of an undulation, at the time T, where the law of vibration is

$$a \sin \frac{2\pi}{\lambda} (vt - x + A),$$

and let (β) represent the state of another undulation at the same time where the law of vibration is

$$b \sin \frac{2\pi}{\lambda} (vt - x + B).$$

If from any point in (α) we measure upwards a distance equal to the elevation of the corresponding point of (β), or measure downwards a distance equal to the depression of the corresponding point of (β), we shall determine the position of the Roman letters. Their distances from the straight line represent the effect of the superposition of the two undulations. This is evidently an undulation of the same kind, and with waves of the same length, as either of the others. But in the instance as we have supposed it, the addition of the undulation (β) to (α) has diminished the maximum vibration of the latter, and has made the maximum to exist at a different part of the line. Thus we see that the magnitude of vibration in an undulation may be diminished by the addition of another undulation transmitted in the same direction. This is a point of great importance, and deserves the reader's attentive consideration.

14. The geometrical figures which we have given are merely illustrations: the conclusion that we have arrived at will be more readily obtained from the algebraic expressions. Adding together the two disturbances

$$a \sin \left\{\frac{2\pi}{\lambda} (vt - x) + A\right\} \quad \text{and} \quad b \sin \left\{\frac{2\pi}{\lambda} (vt - x) + B\right\},$$

we have for the whole disturbance

$$(a \cos A + b \cos B) \sin \left\{\frac{2\pi}{\lambda} (vt - x)\right\}$$

$$+ (a \sin A + b \sin B) \cos \left\{\frac{2\pi}{\lambda} (vt - x)\right\}.$$

This may be put under the form

$$c \sin \left\{\frac{2\pi}{\lambda}(vt - x) + C\right\},$$

if $c \cos C = a \cos A + b \cos B,\quad c \sin C = a \sin A + b \sin B.$

It is very important to remark that the square of c, the new coefficient, is the sum of the squares of the coefficients of

$$\sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\} \text{ and } \cos \left\{\frac{2\pi}{\lambda}(vt - x)\right\}.$$

We shall have occasion frequently to refer to this theorem.

In the present instance,

$$c^2 = a^2 + b^2 + 2ab \cos (A - B):$$

and the quotient of $c \sin C$ by $c \cos C$ gives

$$\tan C = \frac{a \sin A + b \sin B}{a \cos A + b \cos B}.$$

The form of the expression

$$c \sin \left\{\frac{2\pi}{\lambda}(vt - x) + C\right\}$$

shews that the length of a wave is the same as in either of the undulations compounded; but the difference of value of C from A and B shews that the maximum of vibration for a given particle does not generally take place with the same value of t as for either of the undulations compounded. The magnitude of the maximum vibration, which is c or

$$\sqrt{\{a^2 + b^2 + 2ab \cos (A - B)\}},$$

depends on the value of $A - B$: its greatest value is $a + b$, when $A - B = 0$, and its least value is $a \sim b$, when $A \sim B = 180^\circ$. In these two cases C is equal to one at least of the two angles A and B.

Prop. 6. To examine the effects of interference of two equal and similar undulations: and to shew that when one is

$(p+\frac{1}{2})\times$ length of the wave behind the other (p being whole number) they will destroy each other.

15. In the case of equal vibration, $a=b$. The value of c is then

$$\sqrt{\{2a^2+2a^2\cos(A-B)\}}=2a\cos\frac{A-B}{2},$$

$$\text{and }\tan C=\frac{\sin A+\sin B}{\cos A+\cos B}=\tan\frac{A+B}{2},\quad\text{or } C=\frac{A+B}{2}.$$

If $A-B=0$, the value of c is $2a$, and $C=A$: that is, if we add two such undulations as (β) and (ζ) (fig. 2), we shall have an undulation in which the maxima are at the same places, and the maximum vibration is double what it was before. With any other value of $A-B$, c is less than $2a$; and when $B=A\pm 180^\circ$, c is 0, that is, there is no motion whatever. To understand this clearly, we must consider what is meant by the expression

$$a\sin\left\{\frac{2\pi}{\lambda}(vt-x)+B\right\},$$

or, (in this case of destruction of the motion)

$$a\sin\left\{\frac{2\pi}{\lambda}(vt-x)+A\pm\pi\right\}.$$

This is the same as

$$a\sin\left\{\frac{2\pi}{\lambda}\left(vt-x\pm\frac{\lambda}{2}\right)+A\right\}.$$

Now this is exactly the same expression as

$$a\sin\left\{\frac{2\pi}{\lambda}(vt-x)+A\right\},$$

putting $x\mp\frac{\lambda}{2}$ instead of x. That is, the expression for the disturbance in this second undulation, if $B=A\pm 180^\circ$, is the same as that in the first, provided instead of x we take $x\mp\frac{\lambda}{2}$. That is, one of the undulations may be represented by the

same construction as the other, provided we suppose it in advance or in arrear of the other by half the length of a wave. The undulations (β) and (δ), or (γ) and (ϵ) in figures 1 and 2 have this relation to one another. And it will very easily be seen in fig. 2, that if we compound (δ) with (β) by a process similar to that which we used in fig. 3, (13), the elevations of particles in (δ) will correspond to equal depressions in (β), and *vice versâ*, and consequently by their combination the particles will all be brought to their original position. The same will be true after the time $\frac{T}{4}$ when (β) has been changed to (γ), and at the same time (δ) has been changed to (ϵ); and at every other time: and therefore there will be continued rest. Thus we arrive at the extraordinary conclusion that one undulation may be absolutely destroyed by another with waves of the same length transmitted in the same direction, provided that the maxima of vibrations are equal, and that one follows the other by half the length of a wave. Since the retardation of a whole length of a wave, or two whole lengths, &c., produces no alteration in an undulation, it is plain that a retardation of $\frac{3\lambda}{2}$, $\frac{5\lambda}{2}$, &c. will produce the same effect as a retardation of $\frac{\lambda}{2}$; and thus two undulations will destroy each other if the maxima of vibration be the same and the waves be of the same length and transmitted in the same direction; and if one follow the other by $\frac{\lambda}{2}$, or $\frac{3\lambda}{2}$, or $\frac{5\lambda}{2}$, &c.

16. The reader is requested particularly to remark this apparently strange conclusion. It is of the greatest importance in Physical Optics, for the following reason. We shall refer hereafter to experiments which shew that the mixture of two pencils of light will produce darkness. This fact seems to defy any attempt at explanation on the supposition that light is occasioned by the emission of material particles. But in consequence of the conclusion at which we have just arrived, it is perfectly explicable on the supposition that light

consists of a series of waves of either of the kinds mentioned in (2) and (3), transmitted by some medium which pervades space. It is only necessary for perfect agreement that, in the two pencils which mix, the waves of one precede those of the other by spaces which may be represented by

$$\frac{\lambda}{2}, \quad \text{or} \quad \frac{3\lambda}{2}, \quad \text{or} \quad \frac{5\lambda}{2}, \quad \&c.,$$

which is found experimentally to be true. Any other hypothesis, however, from which the same conclusion could be deduced would be, *primâ facie*, equally entitled to our attention.

PROP. 7. To find the result of the interference of any number of waves.

17. We have shewn in (14) the method of compounding the effects of two waves: the effect of several is found in just the same manner. Suppose for instance, the disturbance produced by one undulation was expressed by

$$a \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + A \right\}:$$

that of a second, by

$$b \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + B \right\};$$

that of a third, by

$$e \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + E \right\}, \quad \&c.$$

The sum of these is

$$(a \cos A + b \cos B + e \cos E + \&c.) \sin \left\{ \frac{2\pi}{\lambda} (vt - x) \right\}$$

$$+ (a \sin A + b \sin B + e \sin E + \&c.) \cos \left\{ \frac{2\pi}{\lambda} (vt - x) \right\},$$

which we will call

$$F \sin \left\{ \frac{2\pi}{\lambda} (vt - x) \right\} + G \cos \left\{ \frac{2\pi}{\lambda} (vt - x) \right\}.$$

This, as in (14), may be put under the form

$$c \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + C \right\},$$

if $$F = c \cos C, \quad G = c \sin C;$$

whence $$c = \sqrt{(F^2 + G^2)}, \quad \tan C = \frac{G}{F}.$$

In some cases, where the effects of an indefinitely great number of indefinitely small waves are to be combined, F and G may be found by integration: c and C are then determined by the same process as that just given.

18. It remains now only to notice some cases of undulation not included in those already treated of. One is, that a single wave may be transmitted through a medium (as we know to be true with regard to air), and then our theorems about interference are not true. This however will not come under consideration, as there is reason to think that a single wave in air or in the medium of light would not produce the sensation of sound or colour. We shall generally consider the undulation as a succession of a great (but not infinite) number of waves. Another is, that the magnitude of the maximum vibration of a particle may depend on its situation: for instance, if waves diverge from a center, the vibrations must be more violent in the neighbourhood of that center than at a distance from it. This will be represented by expressing the extent of vibration at any time by

$$\psi(x) \cdot \phi(vt - x),$$

or by the sum of several such functions. Since two successive waves here would not be equal, our theorem about interferences of waves lagging $\frac{\lambda}{2}$, $\frac{3\lambda}{2}$, &c. would not be strictly true: but it is easily conceivable that, at a distance from the center of divergence, the neighbouring waves would be so nearly equal that our expressions would have no sensible error.

19. Another case is, the interference of undulations whose

waves are of different length. We know with respect that the velocity of transmission is the same for waves lengths: and we might expect the same to hold good regard to light. We shall afterwards refer to experi which appear to shew that this is not true. Still wl v be constant or not, it is impossible to unite two terms as

$$a \sin \left\{\frac{2\pi}{\lambda}(vt - x) + A\right\} \text{ and } b \sin \left\{\frac{2\pi}{\lambda'}(v't - x) + B\right\}$$

so as to destroy the original form. We have seen (17 when any number of waves is combined, supposing v a the same for all, their sum contains no trace of the distir of waves in which it originated; and fixing upon an point, or considering x as constant, the vibration is expressed by

$$c \sin \left\{\frac{2\pi v}{\lambda} t + \left(C - \frac{2\pi x}{\lambda}\right)\right\},$$

the same form as that for the vibration of a cycloidal dulum. But the two expressions

$$a \sin \left\{\frac{2\pi v}{\lambda} t + \left(A - \frac{2\pi x}{\lambda}\right)\right\} \text{ and } b \sin \left\{\frac{2\pi v'}{\lambda'} t + \left(B - \frac{2\pi}{\lambda}\right.\right.$$

cannot be combined into that form, unless $\frac{v}{\lambda} = \frac{v'}{\lambda'}$, which is no reason to think true. The consideration therefo waves of different lengths may be kept perfectly separa their ultimate effect will be the same as the sum of all separate effects, without any possibility of their destroyi modifying one another.

20. The reader is requested to attend to the convent signification of the following terms.

By a *wave* we mean all the particles included bet two which are in similar states of displacement and of mc For instance, in any one of the cases (β), (γ), (δ), (ϵ), (fig. 1 or 2, the particles included between b and b' a wave: or those between f' and f'' form a wave: &c.

is easily seen that a wave includes particles in every possible state of displacement and of motion consistent with undulatory vibration.

The *length of a wave* we have explained to be the distance between two particles similarly displaced and moving similarly. The interval, in time, of two waves (that is, the interval between the arrival of two successive waves at the same point), it will be recollected, is the same as the time of vibration of any particle, (5).

By the *phase* of a wave, we shall denote the situation of a particle in a wave, considered as affecting its displacement and motion. For instance, b and b' in fig. 1 or 2, are in *similar phases*, because their displacements are equal, and their motions are also equal. But in (β) fig. 2, b and f are not in the same phase: for though their displacements are equal, their motions are in opposite directions. Similarly f and h are not in the same phase, for their displacements are different though their motions are equal. It is readily seen that particles are in the same phase when the distance between them is a multiple of the length of a wave.

We shall say that particles are in *opposite phases* when the displacement and motion of one are equal and opposite to those of the other. For instance, a and g, or e' and l', in (β), (γ), (δ), (ϵ), or (ζ), of fig. 1 or 2 are in opposite phases. It is easy to perceive that particles are in opposite phases when their distance is an odd multiple of half the length of a wave.

In speaking of waves where the displacement of a particle is represented by

$$a \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + A \right\},$$

we shall consider the arc

$$\left\{ \frac{2\pi}{\lambda} (vt - x) + A \right\}$$

as the measure of the *phase*.

When we consider a wave as extending over space in a direction different from that in which it is transmitted, we shall use the term *front of a wave* to denote a continuous line

passing through all those points which are in the same phase. Thus in fig. 4, suppose that a series of waves of an undulation are passing from the side AB towards CD: and suppose that the line EF passes through a number of points which are simultaneously in the same phase: then EF is a front of a wave. If at the same time GH be another line passing through a number of points which are simultaneously in some other phase, GH is another front of a wave. In considering space of three dimensions it is plain that the front of a wave will generally be a surface.

21. We shall now state a principle of which we shall make extensive use in the calculation of Optical phenomena.

The effect of any wave in disturbing any given point may be found by taking the front of the wave at any given time, dividing it into an indefinite number of small parts, considering the agitation of each of these small parts as the cause of a small wave which will disturb the given point, and finding by summation or integration the aggregate of all the disturbances of the given point produced by the small waves coming from all points of the great wave.

In demonstration of this principle it seems sufficient to say that the agitation of one part at one time is really and truly the cause of the agitation of another part at another time: and that the effect of the great wave, which really is a number of small agitations, will by (12) be the sum of the effects of all the small agitations.

22. A question now arises. What is to limit the waves diverging from each of these small sources of motion? The disturbance spreads generally in a spherical form, so that the front of each little wave is a sphere: are we to suppose the sphere complete, so that each small undulation is propagated backwards as well as forwards?

The following answer appears to be correct, but its application in several cases seems doubtful. The effect of each small wave must be limited by the same considerations which limit the effect of the great wave. Now we know, from the algebraical investigation, that a single wave may be transmitted along a stretched cord, or in air, without being fol

lowed by another*. In this case it is plain that the present agitation of one point causes the future agitation of the points in the direction in which the wave is going, but of none in the direction from which it came. In figure 4 for instance, if a single wave is going from AB towards CD, and if EF be the front of the wave at any time, then we know that the displacement in EF is the cause of future displacement in GH, because in consequence of the existence of this wave there will hereafter be a wave at GH: but we know that the displacement in EF causes no future displacement between EF and AB, because, though the displacement in EF exists, there will hereafter be no wave between AB and EF. If then we divide EF into a great number of parts, we must consider the displacement in each as causing a hemispherical or nearly hemispherical wave, which diverges only before the front of the great wave and not behind it.

APPLICATION OF THIS THEORY TO THE EXPLANATION OF THE PHÆNOMENA OF LIGHT WHICH DO NOT DEPEND ON POLARIZATION.

23. We shall suppose that light is the undulation of a medium called *ether* which pervades all transparent bodies. Respecting the direction of vibration of each particle we shall make no supposition till we treat of polarized light, as the

* The function X which expresses the disturbance may be *discontinuous*, that is, may be expressed by different algebraical forms for different values of $vt-x$, (which we will call w), provided that $\frac{dX}{dw}$ does not alter *per saltum*. For instance $X=0$ while w is less than b: $X=a$ versin $\left\{\frac{2\pi}{\lambda}(w-b)\right\}$ from $w=b$ till $w=b+\lambda$: $X=0$ while w is $>b+\lambda$. This is a discontinuous function expressing a single wave (since the particles are only disturbed when $vt-x$ is $>b<b+\lambda$). And it satisfies the condition just mentioned, since $\frac{dX}{dw}$ is 0 till $w=b$: its value is then $a\frac{2\pi}{\lambda}\sin\left\{\frac{2\pi}{\lambda}(w-b)\right\}$ from $w=b$ to $w=b+\lambda$, that is, it increases gradually from 0 to $a\frac{2\pi}{\lambda}$ and diminishes gradually from $a\frac{2\pi}{\lambda}$ to 0: and when w is $>b+\lambda$, $\frac{dX}{dw}$ is 0.

results of this section are independent of the direction of vibration: to fix the ideas, however, the reader may conceive it to be of the kind represented in fig. 2. We shall suppose that a great number of similar waves follow without interruption, and that the function which expresses the displacement of a particle is

$$a \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + A \right\}.$$

When in our final results we have found the expression

$$c \sin \left\{ \frac{2\pi}{\lambda} vt + C - \frac{2\pi}{\lambda} x \right\}$$

for the displacement of the particles touching a screen or touching the eye, we shall assume the intensity of the light to be represented* by c^2. We shall suppose that the colour of light

* We must take some even power of c to represent the intensity, since the undulation where the vibration is expressed by

$$-c \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + C \right\}$$

differs in no respect from that whose vibration is expressed by

$$+c \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + C \right\},$$

except that it is half the length of a wave before or behind it. The propriety of using the second power may be thus shewn. If two candles giving the same light be placed near each other and shine on the same screen, we say that there is twice as much light as if one only were shining on it: and this may be regarded as the experimental definition of double the quantity of light. Now if the vibration excited by one of these be

$$c \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + C \right\},$$

and that excited by the other be

$$c \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + D \right\},$$

the whole vibration will be the sum of these, or

$$c\sqrt{\{2 + 2 \cos (C - D)\}} \sin \left\{ \frac{2\pi}{\lambda} (vt - x) + E \right\},$$

$$\text{where } \tan E = \frac{\sin C + \sin D}{\cos C + \cos D}.$$

depends on the value of λ, but that λ is always very small: that for extreme red rays it is 0,0000266 inch: less for yellow: still less for green, blue, and indigo successively, and least for the reddish violet rays, for which $\lambda = 0,0000167$ inch. Common white light we shall suppose to consist of a mixture of waves of all lengths intermediate to these.

PROP. 8. A succession of waves, whose fronts are parallel to the screen CD (fig. 5) in which is the comparatively small opening AB, is moving towards the screen: to find the magnitude of vibration on any point G of the semicircle CGD.

24. Let H be the center of AB and of the semicircle, and let $HG = r$, $CHG = \theta$, $HA = b$: divide AB into a great number of small parts, and let the distance of one at Z from H be z, and its breadth δz: then

$$ZG = \sqrt{(r^2 - 2r\cos\theta \, . \, z + z^2)}.$$

When a wave comes to AB, consider separately the parts corresponding to the small divisions of AB. It seems reasonable to suppose that each of these small parts will cause a diverging wave of equal intensity for all values of θ. For if the medium were air, and a rush of particles took place as in fig. 1, from EF, the only effect in the small part under consideration would be to cause a condensation of air: and this would cause a wave of equal intensity for all values of θ. If the vibrations were like those of fig. 2, and perpendicular to the plane of the paper, the same thing appears evident: if

If the lights be represented by the square of the coefficient we have

$$\text{light from a single candle} = c^2$$
$$\text{light from two candles} \quad = 2c^2 + 2c^2 \cos(C - D).$$

The value of $C - D$ may be any whatever, and it is probable that in one second of time, from different systems of waves following each other, $\cos(C - D)$ may have gone through all its values many thousand times. To estimate the effect on the eye, we must take the mean of all its values. As the negative values are equal to the positive ones, the mean of all is 0. Thus we have

$$\text{light from two candles} = 2c^2.$$

By using the second power therefore we obtain a result which agrees with our experimental definition: and the same would appear if instead of two candles we had taken three or any other number. But the agreement will not hold if we take any other power of the coefficient as the representative of the illumination.

parallel to the plane of the paper, it is not so clear wh proportion of intensities would be. The maximum of tion when the wave reached G would, if it followed the laws as in air*, vary nearly as $\frac{1}{ZG}$. Now all the little

* When a wave of air diverges symmetrically through any give angle, if r be the original distance of any particle from the center, distance at the time t, $r+h$ the original distance of a second particle; t distance of the latter at the time t will be

$$r+u+h\left(1+\frac{du}{dr}\right)$$

nearly: and the particles which formerly occupied a volume proporti r^2h now occupy a volume proportional to

$$(r+u)^2h\left(1+\frac{du}{dr}\right),$$

or nearly proportional to

$$(r^2+2ru)\,h\left(1+\frac{du}{dr}\right), \text{ or to } r^2h\left(1+\frac{2u}{r}+\frac{du}{dr}\right).$$

Consequently, if the elasticity be as the m^{th} power of the density, the el

$$\text{original elasticity} \times \frac{1}{\left(1+\frac{2u}{r}+\frac{du}{dr}\right)^m}$$

$$\text{original elasticity} \times \left(1-m\frac{2u}{r}-m\frac{du}{dr}\right) \text{ nearly.}$$

The original elasticity would support in opposition to the force of gr column of air whose height is H. Supposing the section of the column its volume is H, and its mass is $H \times$ mass of volume 1; and the origin ticity, estimated as an accelerating force, is $gH \times$ mass of volume 1: he elasticity in the disturbed state

$$\text{mass of volume } 1 \times gH \times \left(1-m\frac{2u}{r}-m\frac{du}{dr}\right).$$

If then we take a portion of the column as disturbed, whose original k, the excess of the elastic force at one end above that at the other

$$\text{mass of volume } 1 \,.\, mgH \,.\, k \,.\, \frac{d}{dr}\left(\frac{2u}{r}+\frac{du}{dr}\right) \text{ nearly:}$$

and the mass is mass of volume k: hence the equation for the disturba the particles is

$$\frac{d^2u}{dt^2}=mgH\frac{d}{dr}\left(\frac{2u}{r}+\frac{du}{dr}\right), \quad = a^2\frac{d}{dr}\left(\frac{2u}{r}+\frac{du}{dr}\right).$$

To solve this equation, let $u = \ldots$

which originate from the different points of AB are in the same phase: hence we may express the disturbance which the little wave from the part δz produces at G by

$$c\delta z \frac{1}{ZG} \sin \left\{ \frac{2\pi}{\lambda} (vt - ZG) \right\}.$$

then $\frac{d^2y}{dr^2} = 2u + r\frac{du}{dr}$; also $\frac{d^2y}{dt^2} = r\int_r \frac{d^2u}{dt^2}$;

and the equation becomes

$$\frac{d}{dr}\left(\frac{1}{r}\cdot\frac{d^2y}{dt^2}\right) = v^2 \frac{d}{dr}\left(\frac{1}{r}\cdot\frac{d^2y}{dr^2}\right);$$

$$\text{whence } \frac{1}{r}\cdot\frac{d^2y}{dt^2} = \frac{v^2}{r}\cdot\frac{d^2y}{dr^2} + \chi''(t),$$

$$\text{and } \frac{d^2y}{dt^2} - v^2\frac{d^2y}{dr^2} = r\,.\,\chi''(t).$$

Solving this partial differential equation,

$$y = r\,.\,\chi(t) - \phi(vt - r) + \psi(vt + r)$$

$$\text{and } u = \frac{d}{dr}\left(\frac{y}{r}\right) = +\frac{\phi(vt-r)}{r^2} + \frac{\phi'(vt-r)}{r} - \frac{\psi(vt+r)}{r^2} + \frac{\psi'(vt+r)}{r}.$$

If we suppose the wave to travel outwards only,

$$u = \frac{1}{r}\phi'(vt - r) + \frac{1}{r^2}\phi(vt - r).$$

$$\text{If } \phi(vt - r) = -a \cos\left\{\frac{2\pi}{\lambda}(vt - r) + A\right\},$$

$$\phi'(vt - r) \text{ will} = \frac{2\pi a}{\lambda} \sin\left\{\frac{2\pi}{\lambda}(vt - r) + A\right\},$$

and the disturbance

$$= \frac{2\pi a}{\lambda r}\sin\left\{\frac{2\pi}{\lambda}(vt - r) + A\right\} - \frac{a}{r^2}\cos\left\{\frac{2\pi}{\lambda}(vt - r) + A\right\}.$$

This may be put under the form

$$c \sin\left\{\frac{2\pi}{\lambda}(vt - r) + A - C\right\},$$

$$\text{where } c = \sqrt{\left(\frac{4\pi^2 a^2}{\lambda^2 r^2} + \frac{a^2}{r^4}\right)} = \frac{2\pi a}{\lambda r}\sqrt{\left(1 + \frac{\lambda^2}{4\pi^2 r^2}\right)}, \text{ and } \tan C = \frac{\lambda}{2\pi r}.$$

The expression for c shews that at a considerable distance the coefficient will be inversely as r: and as from $r = 0$ to $r = \infty$, C decreases from $\frac{\pi}{2}$ to 0, it appears that the wave is accelerated as it goes on, and ultimately gains a quarter of the length of a wave on the space which it would have described with the uniform velocity v or $\sqrt{(mgH)}$.

Expanding,

$$ZG = r - \cos\theta \,.\, z + \frac{\sin^2\theta}{2r} z^2 + \&c.$$

If z be so small, and r so large, that $\frac{z^2}{2r}$ will never exceed a fraction of λ, (or even if it amounts to several multiples of λ), the terms after the second may be omitted. Then the disturbance produced by one little wave is

$$\frac{c\,\delta z}{ZG} \sin\left\{\frac{2\pi}{\lambda}(vt - r + \cos\theta \,.\, z)\right\}:$$

and the sum of all the disturbances is

$$c\int_z \frac{1}{ZG} \,.\, \sin\left\{\frac{2\pi}{\lambda}(vt - r + \cos\theta \,.\, z)\right\}.$$

In the integration we should produce no sensible error if we put r for ZG, and this makes the sum

$$\frac{c}{r}\int_z \sin\left\{\frac{2\pi}{\lambda}(vt - r + \cos\theta \,.\, z)\right\}.$$

Integrating, it is

$$\frac{-c\lambda}{2\pi r\cos\theta} \cos\left\{\frac{2\pi}{\lambda}(vt - r + \cos\theta \,.\, z)\right\},$$

and taking this from $z = -b$ to $z = +b$, the disturbance at G is

$$\frac{c\lambda}{2\pi r\cos\theta}\left[\cos\left\{\frac{2\pi}{\lambda}(vt - r - \cos\theta \,.\, b)\right\} - \cos\frac{2\pi}{\lambda}\left\{(vt - r + \cos\theta \,.\, b)\right\}\right]$$

$$= \frac{c\lambda}{\pi r\cos\theta} \,.\, \sin\frac{2\pi b\cos\theta}{\lambda} \,.\, \sin\left\{\frac{2\pi}{\lambda}(vt - r)\right\}.$$

This represents a vibration whose maximum is

$$\frac{c\lambda}{\pi r\cos\theta} \,.\, \sin\frac{2\pi b\cos\theta}{\lambda}.$$

25. We shall now proceed to compare the values of this expression for different values of θ.

Case 1. Suppose λ much greater than b. (This will generally be the case with sound, as for all audible sounds λ varies from a few inches to several feet.)

Here $\frac{2\pi b \cos\theta}{\lambda}$ will be a small arc, and will not differ much from its sine: putting the arc for the sine, the maximum of vibration becomes

$$\frac{c\lambda}{\pi r \cos\theta} \cdot \frac{2\pi b \cos\theta}{\lambda}, \quad \text{or } \frac{2cb}{r},$$

which is the same for all values of θ.

Case 2. Suppose λ much smaller than b. (This will generally be the case with light.)

For the part nearly opposite to the entering wave, $\cos\theta$ is very small, and

$$\frac{c\lambda}{\pi r \cos\theta} \cdot \sin\frac{2\pi b \cos\theta}{\lambda} = \frac{2cb}{r}.$$

In other parts the disturbance is 0 when

$$\frac{2\pi b \cos\theta}{\lambda} = \pm\pi, \quad \text{or} = \pm 2\pi, \quad \text{or} = \pm 3\pi, \quad \&c.,$$

that is, when

$$\cos\theta = \pm\frac{\lambda}{2b}, \quad \text{or} = \pm\frac{2\lambda}{2b}, \quad \text{or} = \pm\frac{3\lambda}{2b}, \quad \&c.$$

Hence there is a succession of points in which there is absolute darkness. Of the intermediate parts, the brightest will be found (nearly) by making

$$\sin\frac{2\pi b \cos\theta}{\lambda} = \pm 1:$$

$$\left(\text{omitting the value } \frac{2\pi b \cos\theta}{\lambda} = \pm\frac{\pi}{2}\right):$$

then the maximum of vibration is $\frac{c\lambda}{\pi r \cos\theta}$. Consequently the intensity of light at the brightest part of one of the bright portions is to that of the part nearly opposite to the entering

wave, as $\frac{c^2\lambda^2}{\pi^2 r^2 \cos^2\theta}$ to $\frac{4c^2b^2}{r^2}$, or as λ^2 to $4\pi^2 . b^2 . \cos^2\theta$, or as $\frac{\lambda^2}{4\pi^2 b^2 \cos^2\theta}$ to 1. If $\frac{\lambda^2}{b^2}$ be so small as for light (for instance if $\lambda = 0,00002$ and $b = 0,1$ inch, $\frac{\lambda^2}{b^2} = \frac{1}{25000000}$), it is plain that the value of this ratio will be extremely small when $\cos\theta$ has any sensible value, and we may say without perceptible error that, except nearly opposite to the entering wave, there will be complete darkness all round.

26. If in the investigation we had included the terms depending on z^2, we should merely have had a very small addition to

$$\frac{2\pi}{\lambda}(vt - r + \cos\theta . z),$$

and this addition would not sensibly have altered in value while $\frac{2\pi}{\lambda}(vt - r + \cos\theta . z)$ increased by 2π. It will easily be seen that in Case 1 this would have produced no effect, and in Case 2 the maxima of brightness and the absolute darkness would only have been shifted a little way; their number, relative position, and the intensity of light, remaining sensibly the same*. And if for ZG we had put its more accurate value $r - \cos\theta . z$, the terms added to the expression would not have been sensible.

27. The conclusion in Case 2 may also be obtained thus. In fig. 6 divide AB into a number of equal parts Aa, ab, bc, &c. such that the distance of A from G is less than that of a by $\frac{\lambda}{2}$: that of a less than that of b by $\frac{\lambda}{2}$; and so on. The waves from corresponding parts of Aa and ab are, at starting, in the same phase. Consequently when they reach G, the wave from a part of Aa is in advance of that from the corresponding part

* We shall hereafter consider cases in which these terms are sensible.

of ab by $\frac{\lambda}{2}$, or they are in opposite phases, and therefore, by (15), they destroy each other. Thus every part of Aa destroys a corresponding part of ab, and therefore the whole effect of Ab is 0. Similarly the whole effect of bd is 0; &c. Thus if the number of parts be even, there is no vibration produced at G: if it be odd, there is only the vibration produced by the last of the small parts. But, for the position nearly in front of the wave, all the parts are nearly at the same distance, and the vibration is produced by the added effects of all the small waves coming from every part of AB. If Case 1 be considered in the same way, it appears that the difference of the paths of the waves from different parts of the opening is so small in proportion to the length of a wave, that all when they fall on G may be considered to be in the same phase, and therefore every part of the semicircle is in the same state of vibration.

28. The conclusions at which we have arrived are very important as removing the original objection to the undulatory theory of light. It was objected that if light were produced by an undulation similar to that producing sound, it ought to spread in the same manner as sound: that if light coming from a bright point entered a room by a small hole, it ought (instead of going on straight to illuminate a spot on the opposite side) to spread through the room in the same manner as a sound coming in the same direction and through the same hole. The answer appears in the results of the last investigation: the length of the waves of air is much greater than the aperture, that of the waves of light much less: and the same investigation which shews that in the former case the sound ought to spread equally in all directions, shews that in the latter the light ought to be insensible except nearly in front of the hole. We have reason to think that when sound passes through a very large aperture, or when it is reflected from a large surface (which amounts nearly to the same thing) it is hardly sensible except in front of the opening, or in the direction of reflection.

29. Our conclusion with regard to light is also important as removing one source of doubt in several succeeding investi-

gations. In our ignorance of the law of intensity of the vibrations propagated from a center in different directions, we have supposed the intensity equal in all directions: and yet with this supposition we have found that when the aperture is much larger than λ, there is no sensible illumination except nearly in the direction in which the wave was going before it reached the aperture. The same would be true if the intensity diminished according to some function of the angle made with the original direction of the wave. Since then the illumination is (as far as the senses will be able to judge) nothing, except the obliquity is small, whatever be the function, we may assume that function of any form most convenient, provided that it does not alter rapidly in the neighbourhood of the original direction, and does not increase considerably as the angle of obliquity increases.

30. From the result of this investigation it appears also that the motion of every small part of the wave is perpendicular to the front of the wave. For in fig. 5 that part of the wave which passes through the orifice AB illuminates only that part of the semi-circle which is defined by drawing a straight line perpendicular to the front: and in the same manner if we had covered AB and opened another orifice, we should have found that the only illumination was on the part determined by drawing a straight line through the new orifice perpendicular to the front. In this we see the origin of the idea of rays of light. The reader is particularly requested to observe that this theorem is proved only by the demonstration of the proposition above, and depends entirely on this assumption, that the waves of light move with the same velocity in all directions. We shall hereafter speak of cases in which the motion of the wave is not perpendicular to its front.

It will readily be seen that the whole of this applies as well to the motion of the small parts of a wave whose front is not plane.

Prop. 9. To explain the reflection of light on the undulatory theory.

31. We shall again refer to the motion of sound for an

analogical illustration of this point. In fig. 7 let $ABCD$ be the front of a wave (which for simplicity we suppose plane, every part moving in parallel directions) advancing in the direction BB' or CC'' and meeting the smooth wall $C''B'$. Then it appears from the investigation of sound* that after

* Let x, y, z, be the original co-ordinates of any particle of air: and at the time t let them be $x+X$, $y+Y$, $z+Z$. Then the particle which originally had for co-ordinate $x+\delta x$ will at the time t have

$$x+\delta x+X+\frac{dX}{dx}\delta x \text{ nearly};$$

or the distance between two particles in the direction of x, which was originally δx, is, at the time t, $\delta x\left(1+\frac{dX}{dx}\right)$ nearly. Similarly the distances in the directions of y and z which were originally δy and δz are at the time t,

$$\delta y\left(1+\frac{dY}{dy}\right) \text{ and } \delta z\left(1+\frac{dZ}{dz}\right).$$

Consequently, the air which occupied the rectangular parallelopiped whose sides were δx, δy, δz, now occupies the parallelopiped, nearly rectangular, whose sides are

$$\delta x\left(1+\frac{dX}{dx}\right),\quad \delta y\left(1+\frac{dY}{dy}\right),\quad \delta z\left(1+\frac{dZ}{dz}\right).$$

And if the elasticity (represented by the pressure upon a unit of surface) was originally P, and varied as $(\text{density})^m$, $\left(m \text{ being nearly } \frac{4}{3}\right)$, the elasticity of the air in this parallelopiped is nearly

$$P\left(1-m\frac{dX}{dx}-m\frac{dY}{dy}-m\frac{dZ}{dz}\right).$$

This then is the expression for the elasticity of the air about that point whose co-ordinates were originally x, y, z: the alteration of elasticity being supposed small.

Consequently, at the time t, the elasticity about that point whose co-ordinates were originally $x+h$, y, z, is

$$P\left(1-m\frac{dX}{dx}-m\frac{dY}{dy}-m\frac{dZ}{dz}\right)+P\frac{d}{dx}\left(1-m\frac{dX}{dx}-m\frac{dY}{dy}-m\frac{dZ}{dz}\right)h.$$

And therefore if there is a small parallelopiped whose sides were h, k, l, the excess of pressure which urges it on in the direction of x is

$$-P\frac{d}{dx}\left(1-m\frac{dX}{dx}-m\frac{dY}{dy}-m\frac{dZ}{dz}\right)h\,.\,kl$$

$$=mPhkl\frac{d}{dx}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right).$$

any part of this wave, as BCD, has come in contact with the wall, it will proceed in the direction $C''E$, making with the

And if W were the original weight of the air in volume 1, the weight of this parallelopiped is $W.hkl$. Consequently the acceleration in direction of x is

$$\frac{mPhkl}{Whkl}\cdot\frac{d}{dx}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)\times \text{acceleration produced by gravity}$$

$$=\frac{mP}{W}g\frac{d}{dx}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right);$$

$$\text{or } \frac{d^2(x+X)}{dt^2}=\frac{mP}{W}g\frac{d}{dx}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right).$$

Let H be the height of a homogeneous atmosphere: by which we mean that the pressure P would support a column whose height is H and base 1, weighing W for every unit of volume: that is $P=HW$. Then

$$\frac{d^2(x+X)}{dt^2}, \text{ or } \frac{d^2X}{dt^2}=mgH\frac{d}{dx}\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right):$$

and similar equations hold for y and z. These are the general equations for the small disturbances of air.

These equations cannot be integrated generally: but a number of different integrals can be found, adapted to particular purposes. For instance, putting $mgH=v^2$,

$$X=\frac{x-a}{r^2}\phi'(vt-r)+\frac{x-a}{r^3}\phi(vt-r)$$

$$+\frac{x-a}{r^2}\psi'(vt+r)-\frac{x-a}{r^3}\psi(vt+r)$$

$$\text{where } r=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2};$$

with similar expressions, *mutatis mutandis*, for Y and Z. This is the general expression for spherical waves going to and from the center whose co-ordinates are a, b, c.

$$\text{Again } X=\cos\alpha\,.\,\phi(vt-x\cos\alpha-y\cos\beta-z\cos\gamma),$$

$$Y=\cos\beta\,.\,\phi(vt-x\cos\alpha-y\cos\beta-z\cos\gamma),$$

$$Z=\cos\gamma\,.\,\phi(vt-x\cos\alpha-y\cos\beta-z\cos\gamma),$$

where $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$; $\cos\alpha$, $\cos\beta$, $\cos\gamma$, being positive or negative.

This is the equation for plane waves going in the direction of a line which makes with x, y, and z, the angles α, β, γ.

It is particularly to be remarked that the sum of any number of these solutions may be taken for one solution: and also that the functions may be discontinuous, with the limitation mentioned in the note to (22).

Now suppose the plane wave to be interrupted by a wall, whose equation is $x=c$. Since the particles of air constantly remain in contact with the wall, we must have $X=0$ when $x=c$, whatever be the values of y and z. It is plain that the form above given for X will not satisfy this condition, and that no

wall the same angle as CC'', but on the opposite side of the perpendicular: and the front of the wave will be $B'C'D'$, making with the wall the same angle as BCD, but on the opposite side: the extent of vibration &c. remaining as before. And this appears to justify us sufficiently in the assertion that the waves of light may be reflected in the same manner. With regard to the smoothness of the reflecting surface, all that is necessary is that the elevations or depressions do not exceed a fraction of λ.

32. The following is a more independent method of arriving at the same result, and appears satisfactory. In fig. 8 let ABC be the front of a wave going in the direction of AA'. As soon as each successive small portion of this has reached the surface, we will consider it as causing an agitation in the ether next in contact with the surface, and will suppose that agitation to be the center of a spherical wave, diverging with the same velocity as the plane wave {see the note to (24)}. Let us now consider the state of things when A has reached A'. B has reached B' some time before: and would at this time have arrived at D if not interrupted. Consequently it has diverged into a sphere ab whose radius $= B'D$. C reached the surface at a still longer time previous, and would at this time have reached E: it has therefore diverged into a sphere cd whose radius is CE. The same

single function would do so. But with the addition of another function we may satisfy it. Thus, if

$$\begin{aligned}
X &= \cos\alpha \,.\, \phi(vt - x\cos\alpha - y\cos\beta - z\cos\gamma) \\
&\qquad - \cos\alpha \,.\, \phi\{vt + (x-2c)\cos\alpha - y\cos\beta - z\cos\gamma\}, \\
Y &= \cos\beta \,.\, \phi(vt - x\cos\alpha - y\cos\beta - z\cos\gamma) \\
&\qquad + \cos\beta \,.\, \phi\{vt + (x-2c)\cos\alpha - y\cos\beta - z\cos\gamma\}, \\
Z &= \cos\gamma \,.\, \phi(vt - x\cos\alpha - y\cos\beta - z\cos\gamma) \\
&\qquad + \cos\gamma \,.\, \phi\{vt + (x-2c)\cos\alpha - y\cos\beta - z\cos\gamma).
\end{aligned}$$

The differential equations are satisfied, and the condition $X=0$ when $x=c$ is also satisfied. The first terms of X, Y, Z, taken together, express the original wave, whose direction makes angles α, β, γ, with x, y, z; the second express the new or reflected wave, whose direction makes angles $180^0 - \alpha$, β, and γ with x, y, and z. This shews that the direction of reflection follows the law commonly enounced as the law of reflection. The intensity of the reflected wave is the same as that of the incident wave. This is the theory of oblique echos.

holds for every intermediate point. If now we examine the nature of the front of the grand wave formed by all these little waves, we see that it must be the plane which touches all the spheres, and which evidently makes the same angle with CA' that $A'E$ or AC makes with it, but inclined the opposite way. The motion of the wave, which is perpendicular to this front, makes therefore the same angle after reflection as before.

It must be remarked that this demonstration is in no wise affected if we suppose *all* the spherical waves to be accelerated or retarded by the same quantity. If then we should find occasion hereafter to assert that in some cases the direction of vibration is changed at reflection, or (which amounts to the same) that half the length of a wave must be added to or subtracted from the path after reflection, the demonstration of the law of reflection will not be invalidated.

PROP. 10. To explain the refraction of light on the undulatory theory.

33. We must assume that the waves are transmitted with smaller velocity in glass, water, &c., and in all substances commonly called refracting media, than in what we call vacuum. This assumption appears in the highest degree probable, whether we suppose the vibrations in the refracting media to be vibrations of the same ether, incumbered by its connexion with the particles of the refracting body, or we suppose the vibrations to be vibrations of the particles of the refracting body.

34. Now in fig. 9, let ABC be the front of a wave going in the direction of AA'. As soon as each successive small portion of this wave has reached the surface CA' of the refracting medium, suppose it to cause an agitation in the ether or the particles of the medium at that surface, and consider that agitation to be the center of a wave, which diverges in a spherical form in the medium with a less velocity than the velocity of the plane wave. Consider the state of the particles when A has arrived at A'. B has reached B' some time before: and would have arrived at D if not interrupted by the refracting medium. Consequently it has diverged into

a sphere ab whose radius is less than $B'D$ in the proportion in which the velocity is diminished. Let μ be the number expressing this proportion: then

$$B'b = \frac{1}{\mu} B'D.$$

Similarly C has reached the surface still longer before, and has therefore diverged into a sphere whose radius

$$Cc = \frac{1}{\mu} CE.$$

The same holds for every intermediate point. Now the front of the grand wave formed by all these little waves is evidently the plane which touches all the spheres; and therefore makes with the refracting surface an angle whose sine is $\frac{Cc}{CA'}$. This angle is equal to the angle which the direction of the wave (or the perpendicular to its front) makes with the perpendicular to the refracting surface: it is therefore the angle of refraction. Consequently

$$\sin.\,\text{refraction} = \frac{Cc}{CA'}.$$

Similarly $$\sin.\,\text{incidence} = \frac{CE}{CA'}.$$

Therefore $$\frac{\sin\,\text{refraction}}{\sin\,\text{incidence}} = \frac{Cc}{CE} = \frac{1}{\mu}.$$

35. It is easily seen that a similar demonstration applies when the waves are transmitted in the second medium with greater velocity than in the first: which we suppose to represent the circumstances of light coming out of a refracting medium into vacuum. If in this case, fig. 10, the angle of incidence is so great that Cc (the radius of the wave diverging from C) is greater than CA', that is, if $AA'.\mu$ is greater than CA' or $\frac{AA'}{\sin ACA'}$, or $\sin ACA'$ is greater than $\frac{1}{\mu}$, or sin. incidence greater than $\frac{1}{\mu}$, no plane can be found

which touches all the spheres. There will be no grand wave therefore: and the little waves causing displacements in different directions will very soon destroy each other. Thus there will be no refracted ray. This is a well known law of optics.

It must be remarked that the demonstrations of (32) and (34) are not free from obscurity, for the reason mentioned in (22).

36. There is another phænomenon attendant on refraction which we can explain but vaguely, though it is easily seen that the explanation is not without foundation. The particles of ether next in contact with the glass, (if we suppose glass to be the refracting medium) communicating motion to the denser ether within the glass, may be considered as small bodies striking large ones. Now if they followed the same law as elastic bodies*, a certain motion would be communicated to the large bodies, and the small bodies would lose their original motion and would receive a motion in the *opposite* direction. The motion of the struck bodies causes the refracted wave of which we have just spoken; the motion remaining to the striking bodies will cause a reflected wave in the ether. The magnitude of the reflection will plainly be diminished as the difference between the particles is diminished. Thus refraction is always accompanied by reflection: and the reflection is more feeble as the vibrating media on both sides of the surface approach more nearly to the same state: that is, as the refractive index approaches to 1. This is experimentally true.

37. If, however, the rays are passing from glass to air, we must represent the state of the particles by large bodies striking small ones. The small bodies receive a motion,

* The motions would follow this law, if the particles acted on one another like those of air by condensation or rarefaction of the fluid between them: or in the manner which Fresnel supposes (to be alluded to hereafter): or in any way which makes the force equal at equal distances of the particles. We are not therefore making the forced supposition of particles impinging on each other.

which causes the refracted wave: the large bodies will preserve a part of their motion in the *same* direction, and this will cause a reflected wave. Thus when light passes through glass there will be reflection at both surfaces. But there is this difference between the two reflections: one is caused by a vibration in the same direction as that of the incident ray, and the other by a vibration in the direction opposite to that of the incident ray. We shall find this distinction important in explaining a fundamental experiment (65).

The same thing may be thus shewn. If we suppose a mass of glass to be cracked and the separated parts to be again pressed close together, there will be no more reflection than from the interior of a mass of glass: that is, there will be none at all. Still as there are really two surfaces in contact, each of which separately reflects, we must suppose the reflections to be of such a kind that they destroy each other. Consequently if the vibration from one reflection be in one direction, that from the other reflection must be in the opposite direction.

38. We shall now state an obscurity in the undulatory theory of refraction which has not yet been entirely elucidated, but which does not appear to present any particular difficulty. The index of refraction we have found to be the proportion of the velocities of the waves in vacuum and in the refracting medium. Now it is well known that, experimentally, the refractive index is different for rays of different colours, that is, for waves whose lengths are different. It is evident then that waves whose lengths are different are transmitted with different velocities either in vacuum*, or in the refracting medium, or in both. The difference does not depend on the extent of vibration of each particle, for the refractive index is the same for a bright light as for a feeble one, but merely on the length of the wave, or on the time of vibra-

* If the velocities for different rays were different in vacuum, the aberration of stars (which is inversely as the velocity) would be different for different colours, and every star would appear as a spectrum whose length would be parallel to the direction of the Earth's motion. We know of no reason to think that this is true.

tion*. We are unable to explain this; and one analogy which has guided us in several instances fails here entirely. There is no such difference between the velocities of transmission of long and short waves of air. The sounds of the deepest and the highest bells of a peal are heard at any distance in the same order. Happily another analogy comes in to our aid, derived from undulations in which the vibrations have a much greater similarity than those of air to the vibrations which appear to constitute light. The velocities of waves of water vary with the length of the wave, the long waves always travelling with the greatest speed. In this respect there is a fair analogy between the relations of the velocity to the length of the wave, in glass or other refracting substances (as regards luminiferous waves), and in water (as regards waves of water). But the proportion of the velocity of a long wave of water to a short one is not invariable, but depends upon the depth of the water†. In this may be found an analogy, of a less satisfactory kind, to the change in the relation of the velocity to the length of the wave, on comparing one refractive medium with another. We may remark, however, that if we calculate theoretically the velocity of a

* The difficulty might perhaps be explained thus. We have every reason to think that a part of the velocity of sound depends on the circumstance that the *law of elasticity* of the air is altered by the *instantaneous* development of latent heat on compression, or the contrary effect on expansion. Now if this heat required *time* for its development, the quantity of heat developed would depend on the time during which the particles remained in nearly the same relative state, that is on the time of vibration. Consequently the law of elasticity would be different for different times of vibration, or for different lengths of waves: and therefore the velocity of transmission would be different for waves of different lengths. If we suppose some cause which is put in action by the vibration of the particles to affect in a similar manner the elasticity of the medium of light, and if we conceive the degree of development of that cause to depend on time, we shall have a sufficient explanation of the unequal refrangibility of differently coloured rays.

† Putting k for the depth of the water, λ for the length of a wave, the velocity of the waves is

$$\sqrt{\left(\frac{g\lambda}{2\pi}\cdot\frac{\epsilon^{\frac{2\pi k}{\lambda}}-\epsilon^{-\frac{2\pi k}{\lambda}}}{\epsilon^{\frac{2\pi k}{\lambda}}+\epsilon^{-\frac{2\pi k}{\lambda}}}\right)}.$$

See *Encyclopædia Metropolitana*, *Tides and Waves*, Art. 168.

wave of light, on the supposition that the distance between the particles of ether is a sensible quantity in comparison with the length of a wave, we find that the velocity is different for waves of different lengths. (See the note to Art. 103.) The very close approximation to observed proportions of velocity which has already been attained by calculations of this kind seems to offer good reason for supposing that the explanation will be made complete. On the whole, we regard the slight imperfection of the theory as deserving attention, but not as offering the most trifling difficulty to the general adoption of the theory.

PROP. 11. To explain the course of waves after reflection or refraction.

39. First we must give a construction for determining the front of a wave at any time from knowing its front at any previous time. In fig. 11 let AB be a small part of the curved front of a wave: and draw Aa, Bb, normals to the front, and let $Aa = Bb$. Then by (30), the motion of the part A of the wave is in the direction Aa, and the motion of the part B is in the direction Bb: and their velocities are equal. Consequently after a short time the front of the wave will pass through a, b, and through every other point as c, d, &c., determined by this condition that Cc, Dd, &c. are perpendicular to the original front and are equal to Aa. Now by (30) the motion of the part of the wave at a will be perpendicular to the new front $abcd$. But as $Bb = Aa$, it is easily seen that ab is parallel to AB and is therefore perpendicular to Aa: and therefore the motion of the new front will be in the direction of Aa produced: that is in the same straight line as before. In the same manner it would appear that all its succeeding motion will be in the same straight line. And the velocity will be equal for all points of the wave. Thus we arrive at the following construction. Draw AP, BQ, CR, DS, &c. normals to the original front, and make AP, BQ, CR, DS, &c. equal to the space through which a wave will travel in the time for which the form is required. The front of the wave at that time will be the locus of the points P, Q, R, S. The relation between the old and the new front is therefore this: every line which is normal to one is also

normal to the other. If we confine ourselves to space of two dimensions, this is comprehended in saying that they have the same evolute. This applies equally to the motion of a wave in vacuum or in glass, or in any other refracting medium, provided that the velocity of a wave's motion is equal in all directions.

40. Suppose now (to fix upon a particular case) a plane wave is received on a reflecting paraboloid whose axis is parallel to the direction of the wave's motion, or perpendicular to its front. In fig. 12 let AD be one position of the wave, $A'D'$ a succeeding position, and so on. From (31) and (32) it appears that the front of each small part of the wave makes the same angle with the surface after reflection as before, but on the opposite side of the normal: and that consequently the line representing the direction of the wave's motion, and which is perpendicular to the front, makes the same angle with the normal before and after reflection. As all the lines representing the direction of motion of different points of the wave are parallel to the axis of the paraboloid, those which represent the direction of motion after reflection (by a well known theorem) converge to F the focus. Consequently the form of the wave, which by (39) is the surface to which all these lines are normals, is a spherical surface whose center is F. Thus then at one time $A'D'$ will be the front of the wave: at a later time BC will be the form of that part which is not reflected, and $A''B$, $D''C$, the form of those parts which are reflected, the part incident at A' having been reflected to A'': at a still later time, bc will be the form of the part not reflected, and $A'''B'b$, $D'''C'c$ the form of the reflected parts, the part incident at A' having been reflected to A''', and that incident at B having been reflected to B', &c.: and when the whole has been reflected, all trace of the original form of the wave will be lost, and the existing form will be only a spherical surface of which F is the center. The concave spherical wave goes on towards F, contracting till it passes through that point, when all the different small parts cross, and then they form a diverging spherical wave of which F is the center.

It is easily seen that an explanation of exactly the same

kind applies to the effects of refraction, the velocity of the wave being supposed to be altered in a given ratio as in (33) and (34), and the direction of the motion of each part of the wave being always supposed perpendicular to that part of the front.

41. We have explained the motion of the wave after reflection or refraction as if the terminating edges of the front of the wave did not cause any disturbance beyond the line perpendicular to the front: as if for instance there were a certain disturbance all along the line $A''B$ which afterwards arrived at $A'''B'$ without causing the least disturbance in the ether beyond A'''. This however is not true, and we shall hereafter take into account the effects of the lateral spread of the waves.

42. From the nature of the demonstration it appears that whenever all the small parts of a wave meet each other after reflection or refraction, they have described paths corresponding to equal times. In the case of reflection, this is the same as saying that the whole paths (consisting of the sum of those before and after reflection) must be the same for every point: but in the case of refraction a different statement is necessary. For if the waves move in glass (or other refracting media) with a velocity which is $\frac{1}{\mu} \times$ that in vacuum, then the path in glass, as compared with that in air, is not to be estimated by its length, but by $\mu \times$ its length. And therefore when all the small parts of a wave meet each other after refraction, the sum of the path in vacuum and $\mu \times$ the path in glass is the same for all.

43. This principle may be advantageously applied in the solution of some problems. Suppose for instance it is required to find the form of a refracting surface BP, fig. 13, which shall cause the wave diverging from A to converge to C. The principle above mentioned gives us at once this equation,

$$AP + \mu \, . \, PC = \text{constant}:$$

which is the same as Newton's in the 97th Proposition of the Principia.

44. A *focus* therefore may be defined as the point to which a spherical wave converges, or from which it diverges. It may also be defined as the point at which little waves from all parts of a great wave arrive at the same time. It will readily be seen that our demonstration and our definition include equally real and imaginary foci, in the same manner as the theorems and definitions of common optics.

45. It appears from (40) that the wave, after converging to a point, diverges from it in the same manner as if that point were a center of excitation, or a source of light. In all experiments therefore in which it is wished to produce a series of waves diverging from a point, the image of the Sun's disk, produced by a lens of short focal length, may be used instead of a luminous body.

PROP. 12. A series of waves diverges from a point A, fig. 14, and falls upon two plane mirrors BC, CD, inclined at a very small angle α, and touching each other in the line whose projection on the paper is C: to find the intensity of illumination on different parts of the screen EF where the streams of light reflected from the two mirrors are mixed.

46. Let G be the virtual image (determined by the rules of common optics) of A, produced by reflection at BC: H that produced by reflection at CD. Then instead of supposing the light to have come originally from A, we may without error in our results suppose it to originate in two sources at G and H. For the course of any part of a wave after reflection from BC is just the same as if it had come from G; and the length of its part measured in a straight line from G is the same as the sum of the paths of the incident and reflected ray, since the distance of A and of G from the point of reflection is the same (thus the part of the wave which is incident at N is reflected in the direction NM which is the same as GN produced, by (32) and (39), and the length of its path $AN+MN$ is the same as GM, since $AN = GN$). But that the circumstances may be exactly represented, we must suppose the fictitious wave to start from G at the same time at which the real wave starts from A, and to have the same intensity. Similarly we must suppose the

other wave to start from H at the same time and with the same intensity as that which starts from A; and therefore at the same time and with the same intensity as that which starts from G. The problem is therefore reduced to this: to find the intensity of illumination on the screen when waves start at the same instant and with the same intensity from G and H.

47. Join GH, bisect it in L, and produce LC to meet the screen in O: let M be any point at a small distance from O; $AC = a$: $CO = b$. Since the angle between the mirrors is α, it is easily seen that $GCH = 2\alpha$. And since

$$GC = AC = HC,$$

CL is perpendicular to GH and bisects the angle GCH. Consequently

$$GL = LH = a \sin \alpha: \text{ and } LO = a \cos \alpha + b.$$

Then for the disturbance produced at M by the wave coming from G {taking the same expression as in (8) and (24)} we have

$$\frac{c}{GM} \sin \left\{ \frac{2\pi}{\lambda} (vt - GM + A) \right\}.$$

The variation in the value of GM is so small that without sensible error we may in the coefficient put GO or LO instead of GM; and thus the disturbance produced at M by the wave coming from G is

$$\frac{c}{LO} \sin \left\{ \frac{2\pi}{\lambda} (vt - GM + A) \right\}.$$

Similarly the disturbance produced by the wave coming from H is

$$\frac{c}{LO} \sin \left\{ \frac{2\pi}{\lambda} (vt - HM + B) \right\}.$$

B however must be equal to A, because the waves on leaving G and H respectively are in the same phase at the same time (which is represented by putting 0 for GM and HM, and re-

quires B to be the same as A). Hence the whole disturbance of the ether at M is

$$\frac{c}{LO}\left[\sin\left\{\frac{2\pi}{\lambda}(vt-GM+A)\right\}+\sin\left\{\frac{2\pi}{\lambda}(vt-HM+A)\right\}\right],$$

$$\text{or } \frac{2c}{LO}\cos\left\{\frac{\pi}{\lambda}(GM-HM)\right\}.\sin\left\{\frac{2\pi}{\lambda}\left(vt-\frac{GM+HM}{2}+A\right)\right\},$$

and consequently, by (23), the intensity of the light at M is represented by

$$\frac{4c^2}{LO^2}\cos^2\left\{\frac{\pi}{\lambda}(GM-HM)\right\}.$$

48. Now

$$GM^2=LO^2+(GL+OM)^2=(a\cos\alpha+b)^2+(a\sin\alpha+OM)^2,$$

and consequently

$$GM=a\cos\alpha+b+\tfrac{1}{2}.\frac{(a\sin\alpha+OM)^2}{a\cos\alpha+b} \text{ nearly.}$$

Similarly

$$HM=a\cos\alpha+b+\tfrac{1}{2}.\frac{(a\sin\alpha-OM)^2}{a\cos\alpha+b} \text{ nearly:}$$

therefore

$$GM-HM=\frac{2a\sin\alpha\,.\,OM}{a\cos\alpha+b}=\frac{2a\sin\alpha}{a+b}\,OM \text{ nearly:}$$

and therefore the brightness at M is represented by

$$\frac{4c^2}{(a+b)^2}\cos^2\left(\frac{2\pi}{\lambda}.\frac{a\sin\alpha}{a+b}OM\right).$$

This varies according to the position of M.

(1) Suppose M to coincide with O: $OM=0$: and the brightness is $\dfrac{4c^2}{(a+b)^2}$. This is its greatest value.

(2) Suppose $OM=\pm\dfrac{a+b}{a\sin\alpha}.\dfrac{\lambda}{4}$.

$$\text{Then } \frac{2\pi}{\lambda} \cdot \frac{a \sin \alpha}{a+b} OM = \pm \frac{\pi}{2},$$

and the brightness

$$= \frac{4c^2}{(a+b)^2} \cos^2 \frac{\pi}{2} = 0,$$

or there is absolute blackness.

(3) Suppose $OM = \pm \frac{a+b}{a \sin \alpha} \cdot \frac{2\lambda}{4}$.

$$\text{Then } \frac{2\pi}{\lambda} \cdot \frac{a \sin \alpha}{a+b} OM = \pm \pi,$$

and the brightness

$$= \frac{4c^2}{(a+b)^2} \cos^2 \pi = \frac{4c^2}{(a+b)^2},$$

the same as when $OM = 0$.

(4) Suppose $OM = \pm \frac{a+b}{a \sin \alpha} \cdot \frac{3\lambda}{4}$:

$$\text{then } \frac{2\pi}{\lambda} \cdot \frac{a \sin \alpha}{a+b} OM = \pm \frac{3\pi}{2},$$

and the brightness

$$= \frac{4c^2}{(a+b)^2} \cos^2 \frac{3\pi}{2},$$

or there is darkness.

(5) Generally, if

$OM = \pm 2n \frac{a+b}{a \sin \alpha} \cdot \frac{\lambda}{4}$, n being a whole number,

the brightness has its maximum value $\frac{4c^2}{(a+b)^2}$:

and if $OM = \pm (2n+1) \frac{a+b}{a \sin \alpha} \cdot \frac{\lambda}{4}$,

there is blackness.

Between these values of OM, the brightness has intermediate values.

Thus it appears that there will be a series of points at equal distances along the line IK, at which the illumination is alternately maximum and minimum, beginning at O where it is maximum; and that at the points of minimum illumination the light is actually evanescent. Now considering the screen as extended in the direction perpendicular to the paper, and observing that the investigation which applies to M applies to every point in the line perpendicular to the paper at M, it is easily seen that the appearance on the screen is a series of bars alternately bright and black.

49. We have supposed the plane of reflection to be perpendicular to the edge where the two mirrors touch. If however it had been inclined in any manner, the result would have been precisely the same.

50. We have not yet considered the effect of a mixture of light of different colours in the same pencil (such as exists in white sun-light, and in most kinds of artificial light). According to the suppositions made in (23) this is represented by supposing the light to consist of different series of waves which may or may not be intermixed, the value of λ being different for each different series: and by (19) these different series cannot affect one another, and therefore the effect of each in producing illumination of its peculiar colour is to be considered separately, and the sum of the effects of all the illuminations to be taken afterwards.

51. Now if we examine the expression for the illumination, it will appear that at O the intensity is $\dfrac{4c^2}{(a+b)^2}$ whatever be the value of λ. Consequently, that point is illuminated by each of the differently coloured lights, with four times the quantity of illumination which there would have been if the light from one mirror only had fallen upon it. But there is no other point in similar circumstances. For if we put λ', λ'', λ''', &c. for the lengths of waves of differently coloured lights, λ' being the smallest and corresponding to the blue light, and

c', c'', c''', &c. for the coefficient of displacement, and if we consider the point where $OM = \frac{a+b}{a \sin \alpha} \cdot \frac{\lambda'}{2}$, we find

$$\text{for the intensity of the blue light, } \frac{4c'^2}{(a+b)^2},$$

$$\text{for the intensity of the next kind of light, } \frac{4c''^2}{(a+b)^2} \cos^2 \frac{\pi\lambda'}{\lambda''},$$

$$\text{for the intensity of the third kind of light, } \frac{4c'''^2}{(a+b)^2} \cos^2 \frac{\pi\lambda'}{\lambda'''};$$

and so on. If they had been in the same proportion as in the light reflected from a single mirror, the intensities would have been

$$\frac{4c'^2}{(a+b)^2}, \quad \frac{4c''^2}{(a+b)^2}, \quad \frac{4c'''^2}{(a+b)^2}, \quad \text{\&c.}$$

The different colours therefore are not mixed in the same proportion as in the original light. The same may be shewn of any other point: and thus if the original light be white, no point of the screen will be illuminated with white light except the middle of the central bright bar.

52. The same thing may be thus shewn. The breadth of the bright and dark bars for each colour is proportional to the value of λ for that colour (48). Consequently the bars are narrower for the blue rays than for the green: narrower for the green than for the yellow; &c. by (23). The line passing through the paper at O is however to be the center of a bright bar of each colour. In that line therefore there will be a perfect mixture of all the colours: at a line on each side there will be nearly a total absence of all: but beyond this the red bars will sensibly overshoot the yellow and green and blue bars, and the more as we recede farther from the center. Consequently the bars will be coloured, the bright bars being red on the outside and blue on the inside. And after two or three bars, the outside of the red bars will mingle with the inside of the next blue bars, and there will be no such thing as a black bar. This will continue as we recede from O till the colours become mixed in such a way that it is

impossible to distinguish the bars, and the whole is a mass of tolerably uniform white light. This indistinctness of bars, and ultimately their disappearance, always take place when one of the mixed streams of light has described a path longer than that of the other by several lengths of waves. In general, when white light is used, no bars can be seen where the length of one path exceeds that of the other by ten or twelve times the mean value of λ.

53. The quantity λ, as we have mentioned in (23), is so small that it could not be made sensible to the eye. But $\sin \alpha$ may be made as small as we please, and consequently $\frac{a+b}{a \sin \alpha} \lambda$ may be made large enough to be easily visible to the eye. It is by this and similar means that the lengths of waves for differently coloured light have been measured.

54. The agreement of the facts of experiment with these conclusions from the theory is most complete. And this may be considered as the fundamental experiment on which the undulatory theory is established. It is perfectly certain in this experiment that the mixture of two streams of light, whether white or coloured, does produce black. The bars next the central white are remarkably black: and the dark bars beyond the next bright bars are also very black: and upon intercepting either stream of light all these dark bars become bright. It appears plain that no theory of emission of particles can explain this fact: and it seems difficult to conceive that any theory except that of undulations can explain it.

55. We shall occasionally have to mention the system of bright and dark bars described in (52). We shall generally call them the *fringes of interference.*

56. The reader is particularly requested to observe that, when $a+b$ or the distance of G and H from the screen is given, the breadth of the bars for any given colour is inversely as $a \sin \alpha$ (48), or inversely as GH. And generally, the nearer together are the two sources of waves which interfere, the broader are the fringes of interference.

PROP. 13. A series of waves diverging from a point A, fig. 15, falls upon the prism BCD, each of whose sides BC, CD, makes the small angle α with the third side: to find the intensity of illumination on different parts of the screen EF where the two streams are mixed.

57. The investigation is exactly similar to that of the last proposition, with this difference only. By common Optics,

$AG = AH = CA\ (\mu - 1) \sin\alpha$ nearly, $= (\mu - 1)\ a \sin\alpha$ nearly.

In the former investigation we had $GL = LH = a \sin\alpha$. Consequently where we find $a \sin\alpha$ in the former investigation we may put $(\mu - 1)\, a \sin\alpha$, and we shall have the correct expression for this case. Thus the intensity of illumination

$$= \frac{4c^2}{(a+b)^2} \cos^2 \left\{\frac{2\pi}{\lambda} \cdot \frac{(\mu - 1)\, a \sin\alpha}{a+b}\, OM\right\},$$

and the interval at which the centers of the bright and black bars succeed each other is

$$\frac{a+b}{(\mu - 1)\, a \sin\alpha} \cdot \frac{\lambda}{4}.$$

58. The results are exactly similar to those obtained in (50), (51), (52), (the absolute breadth of the bars being different) with the following exception. The breadth of the bars for different colours does not (as before) depend simply on λ, but on $\dfrac{\lambda}{\mu - 1}$. Now μ varies with λ: it is greatest for the blue rays or those for which λ is least, and less for those for which λ is greater, through all the different colours. Consequently the breadths of the bars formed by the different colours are not in the same proportion as before, but are more unequal. The mixture of colours therefore at the edges of those bars which are a little removed from the central bar is not the same as before; and after a smaller number from the center, the colours of the different bars are mixed with each other (52).

PROP. 14. Suppose that in the experiment of Prop. 12 or 13, Articles (46) to (57), a thin piece of glass PQ is placed in

the path of one of the pencils of light: to find the alteration produced in the fringes of interference.

59. Let T be the thickness of the glass: and consider the case of Prop. 12, Article (46). It is plain that, as in (42), the length of that portion of the path of one pencil which traverses the glass is not to be estimated by its linear measure, but by $\mu \times$ that measure, inasmuch as the motion of the wave is slower in glass than in air by that proportion. We must consider therefore that, instead of describing the space T in air, the wave describes a space equivalent to μT in air; and therefore the effect of the glass is the same as that of lengthening the path by $(\mu - 1) T$. Instead of HM in the expression of (47) we must put

$$HM + (\mu - 1) T:$$

and the intensity of light at M is now

$$\frac{4c^2}{LO^2} \cos^2 \left[\frac{\pi}{\lambda}\{GM - HM - (\mu - 1) T\}\right],$$

which as in (48) is changed to

$$\frac{4c^2}{(a+b)^2} \cos^2 \left\{\frac{2\pi}{\lambda}\left(\frac{a \sin \alpha}{a+b}\, OM - \frac{\mu - 1}{2}\, T\right)\right\}:$$

and the places of maximum brightness are now determined by making

$$\frac{a \sin \alpha}{a+b}\, OM - \frac{\mu - 1}{2}\, T = 0, \quad \text{or} \ = \pm \frac{\lambda}{2}, \quad \text{or} \ = \pm \lambda, \text{ \&c.}:$$

or by making

$$OM = \frac{a+b}{2a \sin \alpha}(\mu - 1)\, T, \quad \text{or} \ = \frac{a+b}{2a \sin \alpha}\{(\mu - 1)\, T \pm \lambda\},$$

$$\text{or} \ = \frac{a+b}{2a \sin \alpha}\{(\mu - 1)\, T \pm 2\lambda\}, \text{ \&c.}$$

60. Now if $\mu - 1$ were the same for rays of all colours, it is evident that these expressions would be precisely the

same as those found for the bright points in (48), increased by a constant

$$\frac{a+b}{2a \sin \alpha}(\mu - 1) T.$$

That is, the whole system of fringes would be shifted towards K, without any other alteration. As $\mu - 1$ is not constant, this is not strictly true: the fringes are shifted, but there is also an alteration of colour arising from the difference of spaces (not even proportional to the breadth of the bars) through which the differently coloured bars are shifted.

61. It will readily be imagined that if a piece of common glass were interposed, the lengths

$$GM \text{ and } HM + (\mu - 1) T$$

would, on account of the exceeding smallness of λ, differ in every point between I and K by many multiples of λ, and therefore (52) no fringes would be visible. The experiment may however be successfully performed by taking two pieces of glass from the same plate, whose difference of thickness will be very small, and placing one in the path of one pencil, and the other in the path of the other. But it may be better performed by taking a pretty uniform piece of glass, cutting it across the middle, and holding one half perpendicular to the path of one pencil, and the other half inclined to the path of the other. It is evident that the obliquity of passage produces the same effect as the use of a thicker piece of glass: and by gentle inclination the difference of paths may be made as small as we please.

62. The difference of paths is to be calculated thus. In fig. 16 let $WXYZ$ be the path of a portion of the wave perpendicular to one half, and $RSTV$ the path of another portion (which for simplicity we suppose moving in a parallel direction) through the other half whose angle of inclination is β. Let T' be the thickness. From T draw Ts perpendicular to RS produced. Since the front of the wave in air, when the portion in question is incident at S, is perpendicular to RS at the point S; and since, when the portion has reached T, the front of the wave in air is perpendicular to

TV at the point T; it is plain that the wave has advanced in the direction perpendicular to its front only through the space Ss. But the time which has been occupied in this progress is the time of describing ST in glass or $\mu . ST$ in air. Consequently the retardation (measured by the space which the wave would have advanced further, if it had moved in air) is

$$\mu . ST - Ss.$$

And the retardation of the portion incident at X is

$$(\mu - 1)\,XY \text{ or } (\mu - 1)\,T'.$$

Therefore the upper pencil is more retarded than the lower by

$$\mu . ST - Ss - (\mu - 1)\,T'.$$

The angle of incidence is β: and if γ be the angle of refraction,

$$ST = \frac{T'}{\cos\gamma}, \text{ and } Ss = \frac{T' . \cos(\beta - \gamma)}{\cos\gamma}.$$

$$\text{Also } \mu = \frac{\sin\beta}{\sin\gamma};$$

therefore the retardation

$$= T'\left\{\frac{\sin\beta}{\sin\gamma\cos\gamma} - \frac{\cos(\beta-\gamma)}{\cos\gamma} - \frac{\sin\beta}{\sin\gamma} + 1\right\}$$

$$= 2T'\left(\sin^2\frac{\beta}{2} - \mu . \sin^2\frac{\gamma}{2}\right).$$

If β be small,

$$\sin\frac{\gamma}{2} = \frac{1}{\mu}\sin\frac{\beta}{2} \text{ very nearly,}$$

and the retardation

$$= 2T'\left(1 - \frac{1}{\mu}\right)\sin^2\frac{\beta}{2} \text{ nearly.}$$

This is to be substituted for $(\mu - 1)\,T$ in the expressions of (59) and (60); and the resulting quantity

$$\frac{a+b}{a \sin \alpha} T' \left(1 - \frac{1}{\mu}\right) \sin^2 \frac{\beta}{2}$$

measures the shift of the central bar towards the side on which is the inclined glass.

63. These conclusions are fully supported by experiment: and this is important as establishing one of the fundamental points of the undulatory theory of Optics, namely that light moves more slowly in glass than in air. The whole of the investigation of the last proposition depends on this assumption.

PROP. 15. A series of waves is incident upon two plates of glass separated by a very small interval (fig. 17); part of the light is reflected at the lower surface of the first glass and part at the upper surface of the second glass: and these portions interfere: to find the intensity of the mixture.

64. Let AB be the path of one portion which is refracted in the direction BC, and of which one part is reflected in the direction CD, while another part is refracted at C and falls on the second plate at E, is partially reflected to F, and partially refracted in the direction FG parallel to CD. Draw FD perpendicular to CD. Then the path which one wave has described in going from C to D, measured by the equivalent path in vacuum, is $\mu . CD$: while that which the other has described in going from C to F (where its front has the same position as the front of that which has reached D) is

$$CE + EF.$$

The excess of the latter above the former is

$$CE + EF - \mu . CD.$$

Let D be the distance of the plates, γ the angle of incidence at C, β the angle of refraction. Then

$$CE + EF = \frac{2D}{\cos \beta}:$$

and $CD = FC . \sin \gamma = 2D . \tan \beta . \sin \gamma;$

$$\text{also } \mu = \frac{\sin\beta}{\sin\gamma};$$

therefore the excess

$$= \frac{2D}{\cos\beta} - \frac{2D\sin^2\beta}{\cos\beta} = 2D\cos\beta.$$

If then the extent of vibration in the light reflected from C be

$$A\sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

where the distance x is measured by the equivalent path in air; then the extent of vibration in the wave reflected from E will be represented by

$$B\sin\left\{\frac{2\pi}{\lambda}(vt - x - 2D\cos\beta)\right\};$$

and the whole intensity will be the intensity of the light in which the displacement of a particle is represented by the sum of these quantities. It must be recollected that by the reasoning in (37) we are entitled to suppose that the signs of A and B are different.

65. We have here however omitted the consideration of that part of the light which is reflected from F to H, again partially reflected at H and partially refracted at K: and the other parts successively reflected. It is plain that (putting V for $2D\cos\beta$) the part refracted at K will be retarded $2V$: that at the next point $3V$: and so on. Now suppose that when light goes from glass to air, the incident vibration being

$$a\sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

the reflected vibration is

$$b\,.\,a\sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

and the refracted vibration

$$c\,.\,a\sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

and suppose that when light goes from air to glass, the coefficient is multiplied by e for the reflected vibration, and by f for the refracted vibration. Then if the coefficient for the light passing in the direction BC is a, that for the vibration reflected at C will be ab: that for the vibration refracted at F, $acef$: that for the vibration refracted at K, ace^3f: and so on. Thus the whole vibration is

$$ab \sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\} + acef\left[\sin\left\{\frac{2\pi}{\lambda}(vt-x-V)\right\}\right.$$

$$\left. + e^2 \sin\left\{\frac{2\pi}{\lambda}(vt-x-2V)\right\} + e^4 \sin\left\{\frac{2\pi}{\lambda}(vt-x-3V)\right\} + \&c.\right]$$

$$= a\left[b \sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\} + cef\frac{\sin\left\{\frac{2\pi}{\lambda}(vt-x-V)\right\} - e^2 \sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\}}{1-2e^2\cos\left(\frac{2\pi}{\lambda}V\right)+e^4}\right].$$

We shall anticipate so much of succeeding investigations (see Art. 128 and 129) as to state that, whether the vibrations are in or perpendicular to the plane of incidence*, there is reason to think that

$$\frac{c}{b} = -1, \text{ and } cf = 1 - e^2.$$

Using these equations to simplify the expression; resolving it into the form

$$F \sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\} + G \cos\left\{\frac{2\pi}{\lambda}(vt-x)\right\}$$

as in (17); and, as in (17) and (23), taking $F^2 + G^2$ to represent the intensity, we find for the brightness of the reflected light

$$\frac{4a^2e^2\sin^2\left(\frac{\pi}{\lambda}V\right)}{1-2e^2\cos\left(\frac{2\pi}{\lambda}V\right)+e^4}, \text{ or } \frac{4a^2e^2\sin^2\left(\frac{2\pi}{\lambda}D\cos\beta\right)}{(1-e^2)^2+4e^2\sin^2\left(\frac{2\pi}{\lambda}D\cos\beta\right)}.$$

* When there are vibrations of both these kinds, it is necessary to calculate the illumination from each, and to take their sum.

66. The supposition that we have made is that of a thin plate of air or vacuum inclosed between plates of glass, or mica, &c. But it is plain that the investigations apply in every respect to a thin plate of fluid with air on both sides: as for instance a soap-bubble. To examine particular cases,

(1) If $D=0$, the intensity $=0$ whatever be the value of λ. Thus it is found that where plates of glass &c., are absolutely in contact or very nearly so, there is no reflection: and when a soap-bubble has arrived at its thinnest state, just before bursting, the upper part appears perfectly black.

(2) The intensity is also 0 if

$$D\cos\beta=\frac{\lambda}{2},\ \lambda,\ \frac{3\lambda}{2},\ \&c.$$

But when light of different colours is mixed, it will be impossible to make the light of all the different colours vanish with the same value of D, and thus no value of D will produce perfect blackness.

(3) If $D\cos\beta=\dfrac{\lambda}{4}$, and if we take the value of λ corresponding to the mean rays (as the green-yellow), the intensity of light in the different colours will be nearly in the same proportion as in the incident light, or the reflected light will be nearly white. But this will not take place on increasing the value of D, or the reflected light will be coloured; till D is become so large that for a great number of different kinds of light, corresponding to very small differences of λ, $\dfrac{4D\cos\beta}{\lambda}$ has the values of successive odd numbers; the different kinds of light will then be mixed in nearly equal proportions, and the mixture will be white.

PROP. 16. In the circumstances of the last proposition, to find the intensity of the light refracted into the second plate.

67. It is readily seen that the coefficient of the vibration refracted at E is $a\,.\,cf$: that of the vibration refracted at H is $a\,.\,ce^2f$: and so on. Also the wave entering at H is behind that which entered at E by the same quantity V as before. Hence the sum of the vibrations will be

$$a\,.\,cf\left[\sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\}+e^2\sin\left\{\frac{2\pi}{\lambda}(vt-x-V)\right\}+\&\text{c.}\right]$$

$$=a\,.\,cf\frac{\sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\}-e^2\sin\left\{\frac{2\pi}{\lambda}(vt-x+V)\right\}}{1-2e^2\cos\left(\frac{2\pi}{\lambda}V\right)+e^4}.$$

Treating this in the same manner as in (65), the intensity of light is found to be

$$\frac{a^2(1-e^2)^2}{(1-e^2)^2+4e^2\sin^2\frac{\pi V}{\lambda}}.$$

68. The proportional variations of this expression are much smaller than those of the expression of (65); its greatest value being a^2, and its least $\frac{a^2(1-e^2)^2}{(1+e^2)^2}$. The absolute variations are however exactly the same: and in fact the sum of the two expressions is always $=a^2$. This is expressed by saying that one of the intensities is *complementary* to the other. This relation spares us the necessity of examining every particular case of the value of D. If for any particular value of D the expression of (65) is maximum for any particular colour, that of (67) is minimum for the same colour: and so on. Thus if for some value of D the expression of (65) gave maximum intensity of red light, less of yellow, the mean intensity of green, less of blue, and nothing of violet (the mixture of which would produce a rich yellow): then the expression of (67) would give the minimum intensity of red light, more of yellow, the mean intensity of green, more of blue, and the maximum of violet (the mixture of which would produce a greenish blue diluted with much white). It is to be remarked that in the case of transmitted light the colours can never be so vivid as in reflected light, because none of the

colours ever wholly disappears, as no values of D and λ will make the expression of (67) $=0$.

PROP. 17. Two glass prisms, right-angled or nearly so, (fig. 18) are placed with their hypotenusal sides nearly in contact: light is incident in such a manner that the angle of internal incidence at the hypotenusal side is nearly equal to the angle of total reflection: part of the light is reflected through the first prism, and part is refracted through the second: to find the expressions for the intensities.

69. The investigations and results are exactly the same as those of Prop. 15 and 16: but this case deserves a particular consideration for the following reason. In this case there is no difficulty (which there is in the others) in making the angle of incidence approach as near as we please to the angle of total reflection, and consequently no difficulty in making β (which is the angle of refraction from the first prism into air) as nearly $= 90°$ as we please, or $\cos\beta$ as small as we please. Consequently $D\cos\beta$ may be made extremely small without making D very small. Now if $D\cos\beta$ in Prop. 15 and 16 were *moderately* small $\left(\text{as for instance } \frac{1}{1000} \text{ inch}\right)$, we might find about twenty different colours of light dividing the colours from red to violet by tolerably equal shades, each of which, in consequence of the difference of their values of λ, would make

$$\sin^2\left(\frac{2\pi}{\lambda} D\cos\beta\right) = 1:$$

and between these colours, the expression would have all its changes of value. The mixture of light would therefore be produced by taking parcels from all the various shades of colour, and mixing them in the same proportion as in common light; and therefore would be nearly white. But when $D\cos\beta$ in this proposition is *extremely* small (for instance less than any value of λ, or not many times greater), not one colour perhaps, or not more than one or two, can be found, for which

$$\sin^2\left(\frac{2\pi}{\lambda} D\cos\beta\right) = 1:$$

and thus there will be an excess of some colours, and the light will be strongly coloured.

70. There is also another reason. By a small change of the angle of incidence we produce a small change in γ; and

$$\text{as } \frac{d\beta}{d\gamma} = \frac{\tan\beta}{\tan\gamma},$$

this produces a great change in β (β being nearly $= 90°$). Consequently the change in $D\cos\beta$ is considerable: and the expression for the intensity of light will be varied much. If then the light of the clouds fall in different directions on this combination of prisms, or if the sun-light be made (by a lens) to fall on it in different directions, the light both reflected and transmitted will form on a screen bands of light. As the position and breadth of these bands are different for every different colour, the mixture forms a very splendid series of coloured bands, in which the succession of colours differs from that produced by almost every other phenomenon of interferences. The same effect may be seen as well if the combination of prisms be held to the eye: when the light coming in different directions to the eye will exhibit the bands in great perfection.

PROP. 18. Two convex lenses of small curvature, or a convex lens and plain glass (fig. 19), are placed in contact: to find the intensity of the light reflected and transmitted at any point M.

71. The two surfaces at M will be so nearly parallel that we may without sensible error consider them as parallel*: and therefore the investigations of Prop. 15 and 16 apply. It is only necessary to find an expression for D in terms of OM, O being the point where the lenses are in contact. Let

* As we shall suppose in the investigation that the separation of the two surfaces at M is but a small multiple of λ, it is evident that for the points immediately about M the defect from parallelism will produce an error amounting only to a very small fraction of λ: and therefore the small waves in (32) will have their effects added together in the direction in which light is reflected from one of the surfaces, nearly in the same degree as in the direction in which it is reflected from the other surface.

r be the radius of the lower surface of the upper lens: r' that of the upper surface of the lower lens. Then D or the separation at M is the sum of the versed sines of two circles whose radii are r and r', to the arcs whose chord is OM: and therefore

$$D = \frac{OM^2}{2r} + \frac{OM^2}{2r'} \text{ nearly} = OM^2\left(\frac{1}{2r} + \frac{1}{2r'}\right).$$

The intensity of reflected light (65) will therefore be

$$\frac{4a^2e^2\sin^2\left(\frac{\pi}{\lambda}V\right)}{(1-e^2)^2 + 4e^2\sin^2\left(\frac{\pi}{\lambda}V\right)},$$

and that of the transmitted light (67) will be

$$\frac{a^2(1-e^2)^2}{(1-e^2)^2 + 4e^2\sin^2\left(\frac{\pi}{\lambda}V\right)},$$

$$\text{where } V = OM^2 . \cos\beta . \left(\frac{1}{r} + \frac{1}{r'}\right).$$

(1) The reflected light vanishes when

$$V = 0, \quad \text{or} = \lambda, \quad \text{or} = 2\lambda, \ \&c.,$$

or when

$$OM^2 = 0, \text{ or } = \frac{\lambda\sec\beta}{\frac{1}{r}+\frac{1}{r'}}, \quad \text{or} = \frac{2\lambda\sec\beta}{\frac{1}{r}+\frac{1}{r'}}, \quad \text{or} = \frac{3\lambda\sec\beta}{\frac{1}{r}+\frac{1}{r'}}, \ \&c.$$

Consequently for any particular colour there will be a dark spot at O and a series of dark rings of which O is the center, and the squares of whose diameters are in the proportion of 1, 2, 3, &c.

(2) The most brilliant light is reflected when

$$OM^2 = \frac{\lambda\sec\beta}{2\left(\frac{1}{r}+\frac{1}{r'}\right)}, \quad \text{or} = \frac{3\lambda\sec\beta}{2\left(\frac{1}{r}+\frac{1}{r'}\right)}, \quad \text{or} = \frac{5\lambda\sec\beta}{2\left(\frac{1}{r}+\frac{1}{r'}\right)}, \ \&c.$$

Consequently between the dark rings there are bright rings, the squares of the diameters of whose most brilliant parts are in the proportion of

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \text{\&c.}$$

(3) The diameters of these rings are *cæteris paribus* as $\sqrt{(\sec \beta)}$. Consequently on inclining the incident ray, or on depressing the position of the eye by which they are viewed, the diameters of the rings increase.

(4) For differently coloured rays, the diameters of the rings vary as $\sqrt{(\lambda)}$. The different colours which are mixed in white light produce therefore a series of rings whose diameters are different, and which overlapping each other produce a series of colours analogous to those mentioned in (52). The colours at last become mixed to such a degree that no traces of rings are visible.

(5) The diameters of the rings vary, *cæteris paribus*, as

$$\frac{1}{\sqrt{\left(\frac{1}{r}+\frac{1}{r'}\right)}}.$$

To make the rings large, therefore, $\frac{1}{r}+\frac{1}{r'}$ must be small, or r and r' must be large. If the lower glass be plane, or $\frac{1}{r'}=0$, the diameters of the rings vary as $\sqrt{(r)}$.

(6) The transmitted light, just as in (68), produces rings complementary to those produced by the reflected light. The center therefore is bright, and is surrounded by a dark ring, which is followed by bright and dark rings alternately, which soon become coloured and finally cease to be visible. The diameter of each ring is the same as that of the ring of opposite character produced by reflected light.

72. These are commonly called *Newton's rings*, from the circumstance that the measures which suggested the most important part of Sir Isaac Newton's theory of light, and which have served in a great degree for the foundation of all theories, were made by him on those rings. The colours of the successive rings, arising from the different mixtures of all the colours producing white light, are commonly called *Newton's scale of colours*. In describing them it is usual to describe them by the number of the ring (including the central spot in the reckoning) in which they occur. For instance, the blue of the second order is not the blue which surrounds the central black spot, but the blue which surrounds the first black ring. This scale is valuable, as giving us an invariable series of colours which can at any time be produced without difficulty. The colours described in (52) as resulting from the experiment of Prop. 12 (46) would do as well, but the adjustment of the apparatus for that experiment is much more troublesome.

Prop. 19. Light diverging from a center A (fig. 20), is allowed to pass through a small aperture BC: to find the illumination on different points of the screen DE.

73. Suppose the wave diverging from A to proceed in a spherical form till it reaches CB: there suppose every part of it (within the limits of the aperture) to be the origin of a little wave proportional in intensity to the superficial extent of that part. By the principle of (21), the sum of the disturbances which each of these produces at M is to be taken for the whole disturbance there: and this being found, the intensity of light will be found as in the preceding problems. Draw AO perpendicular to the screen, and consider it as the axis of z, A being the origin: let x be in the plane of the paper perpendicular to AO, and y perpendicular to the paper. Let $AB = a$, $AO = a + b$: (then b is very nearly the distance of the screen from the aperture:) let x, y, and z be co-ordinates of any point P of the wave, and p and q co-ordinates of any point M of the screen in the directions of x and y (that in the direction of z being $a + b$). Suppose the front of the wave divided by lines perpendicular to the paper into narrow parallelograms, x and $x + \delta x$ being the co-ordinates of two of

these lines: and suppose the parallelogram whose breadth is δx to be divided into small parallelograms by lines parallel to the paper, the co-ordinates of two lines being y and $y+\delta y$, or the surface of the small parallelogram formed by the intersection of these with the others being $\delta x \,.\, \delta y$. Then the displacement which the little wave originating at this surface would cause at M will be represented by

$$\delta x \,.\, \delta y \,.\, \sin\left\{\frac{2\pi}{\lambda}(vt - PM)\right\}.$$

$$\text{Now } PM^2 = (p-x)^2 + (q-y)^2 + (a+b-z)^2,$$

which, since $x^2+y^2+z^2=a^2$,

$$\text{is } = (a+b)^2 + a^2 + p^2 + q^2 - 2px - 2qy - 2(a+b)z.$$

$$\text{But } z = \sqrt{(a^2-x^2-y^2)} = a - \frac{x^2}{2a} - \frac{y^2}{2a} \text{ nearly},$$

as x and y are supposed to be small even at their greatest values: therefore

$$-2(a+b)z = -2a^2 - 2ab + \frac{a+b}{a}x^2 + \frac{a+b}{a}y^2:$$

$$\text{and } PM^2 = b^2 + p^2 + q^2 + \frac{a+b}{a}x^2 - 2px + \frac{a+b}{a}y^2 - 2qy,$$

whence

$$PM = b + \frac{p^2+q^2}{2b} + \frac{a+b}{2ab}x^2 - \frac{p}{b}x + \frac{a+b}{2ab}y^2 - \frac{q}{b}y \text{ nearly},$$

$$\text{or } = b + \frac{p^2+q^2}{2(a+b)} + \frac{a+b}{2ab}\left(x - \frac{ap}{a+b}\right)^2$$

$$+ \frac{a+b}{2ab}\left(y - \frac{aq}{a+b}\right)^2.$$

Put B for $b + \dfrac{p^2+q^2}{2(a+b)}$;

then the displacement caused by the little wave is

$$\delta x.\delta y.\sin\left[\frac{2\pi}{\lambda}\left\{vt-B-\frac{a+b}{2ab}\left(x-\frac{ap}{a+b}\right)^2-\frac{a+b}{2ab}\left(y-\frac{aq}{a+b}\right)^2\right\}\right]$$

$$=\delta x.\delta y.\sin\left\{\frac{2\pi}{\lambda}(vt-B)\right\}.\cos\left[\frac{\pi}{\lambda}.\frac{a+b}{ab}\left\{\left(x-\frac{ap}{a+b}\right)^2+\left(y-\frac{aq}{a+b}\right)^2\right\}\right]$$

$$-\delta x.\delta y.\cos\left\{\frac{2\pi}{\lambda}(vt-B)\right\}.\sin\left[\frac{\pi}{\lambda}.\frac{a+b}{ab}\left\{\left(x-\frac{ap}{a+b}\right)^2+\left(y-\frac{aq}{a+b}\right)^2\right\}\right].$$

Call this $\delta x.\delta y.W$: and let v be the sum of all the disturbances for the slice between the lines whose ordinates are x and $x+\delta x$: w that for the whole surface. Since v increases by $\delta x.W.\delta y$ upon increasing y by δy,

$$\frac{dv}{dy} = \text{ultimate value of } \frac{\delta v}{\delta y} = \delta x.W,$$

$$\text{whence } v = \delta x \textstyle\int_y W.$$

This integration is to be performed, and the limits of the values of y for the integral will be expressed in terms of x from a knowledge of the shape of the aperture. The effect of the narrow parallelogram being v or $\delta x \int_y W$, it is found in the same manner that the effect of the whole aperture is $\int_x \int_y W$. As $\sin\frac{2\pi}{\lambda}(vt-B)$ and $\cos\frac{2\pi}{\lambda}(vt-B)$ are not concerned in the integration, the whole displacement may be expressed thus:

$$\sin\frac{2\pi}{\lambda}(vt-B)\int_x\int_y\cos\left[\frac{\pi}{\lambda}.\frac{a+b}{ab}\left\{\left(x-\frac{ap}{a+b}\right)^2+\left(y-\frac{aq}{a+b}\right)^2\right\}\right]$$

$$-\cos\frac{2\pi}{\lambda}(vt-B)\int_x\int_y\sin\left[\frac{\pi}{\lambda}.\frac{a+b}{ab}\left\{\left(x-\frac{ap}{a+b}\right)^2+\left(y-\frac{aq}{a+b}\right)^2\right\}\right].$$

Let the integrals be E and F: then the whole displacement

$$= E\sin\frac{2\pi}{\lambda}(vt-B) - F\cos\frac{2\pi}{\lambda}(vt-B),$$

and hence {as in (14), (17), and (23)} the illumination is represented by E^2+F^2.

74. Now

$$E = \int_x \int_y \left[\cos\left\{\frac{\pi}{\lambda} \cdot \frac{a+b}{ab}\left(x - \frac{ap}{a+b}\right)^2\right\} \times \cos\left\{\frac{\pi}{\lambda} \cdot \frac{a+b}{ab}\left(y - \frac{aq}{a+b}\right)^2\right\} \right.$$

$$\left. - \sin\left\{\frac{\pi}{\lambda} \cdot \frac{a+b}{ab}\left(x - \frac{ap}{a+b}\right)^2\right\} \times \sin\left\{\frac{\pi}{\lambda} \cdot \frac{a+b}{ab}\left(y - \frac{aq}{a+b}\right)^2\right\} \right];$$

proceeding therefore according to the precepts of (73), the first thing to be done is to find

$$\int_y \cos\left\{\frac{\pi}{\lambda} \cdot \frac{a+b}{ab}\left(y - \frac{aq}{a+b}\right)^2\right\} \text{ and } \int_y \sin\left\{\frac{\pi}{\lambda} \cdot \frac{a+b}{ab}\left(y - \frac{aq}{a+b}\right)^2\right\}.$$

These integrals cannot generally be found in terms of y. Tables however have been formed* expressing the numerical value of the integrals,

$$\int_s \cos\frac{\pi s^2}{2} \text{ and } \int_s \sin\frac{\pi s^2}{2}$$

for different values of s, and there is no difficulty in applying them to the numerical expression of the *first* integral in our problem. But as the integral thus found is given in numbers, and not in an expression involving y and which can be

* Suppose U to be the function of unknown form which is the integral of S a given function of s. Then the values of U corresponding to $s - h$ and $s + h$ are

$$U - \frac{dU}{ds}h + \frac{d^2U}{ds^2}\cdot\frac{h^2}{2} - \frac{d^3U}{ds^3}\cdot\frac{h^3}{2\,.\,3} + \&c.$$

$$\text{and } U + \frac{dU}{ds}h + \frac{d^2U}{ds^2}\cdot\frac{h^2}{2} + \frac{d^3U}{ds^3}\cdot\frac{h^3}{2\,.\,3} + \&c.$$

and therefore the value of the integral from $s - h$ to $s + h$ is

$$2\frac{dU}{ds}h + 2\frac{d^3U}{ds^3}\cdot\frac{h^3}{2\,.\,3} + \&c.$$

$$\text{or } 2Sh + 2\frac{d^2S}{ds^2}\cdot\frac{h^3}{2\,.\,3} + \&c.$$

which can be easily calculated: and by taking h small enough, one or two terms will be sufficient. In this way the values of the integral can be computed for successive limits, and the sum will be the integral up to any given value of s. A table of $\int_s \cos\frac{\pi s^2}{2}$ and $\int_s \sin\frac{\pi s^2}{2}$ will be given at the end of this Tract.

expressed in terms of x, it is seldom possible to perform the second integration.

75. In certain cases however this may be effected. For example, suppose the aperture to be a parallelogram whose sides are $2e$ and $2f$ in the direction of x and y, the line AO passing through its center. Let

$$\frac{a+b}{\lambda ab}\left(y-\frac{aq}{a+b}\right)^2=\frac{s^2}{2};$$

$$\therefore\ y-\frac{aq}{a+b}=s\sqrt{\frac{\lambda ab}{2(a+b)}},\ \text{and}\ \frac{dy}{ds}=\sqrt{\frac{\lambda ab}{2(a+b)}};$$

$$\therefore\ \int_y \cos\left\{\frac{\pi}{\lambda}\cdot\frac{a+b}{ab}\left(y-\frac{aq}{a+b}\right)^2\right\}=\sqrt{\frac{\lambda ab}{2(a+b)}}\int_s \cos\frac{\pi s^2}{2};$$

and this is to be taken from $y=-f$ to $y=+f$,

or

$$\text{from}\ s=-\sqrt{\left\{\frac{2(a+b)}{\lambda ab}\right\}}\left(f+\frac{aq}{a+b}\right)$$

$$\text{to}\ s=\sqrt{\left\{\frac{2(a+b)}{\lambda ab}\right\}}\left(f-\frac{aq}{a+b}\right).$$

This will be the sum of the two numbers, in the table of $\int_s \cos\frac{\pi s^2}{2}$, corresponding to

$$s=\sqrt{\left\{\frac{2(a+b)}{\lambda ab}\right\}}\left(f+\frac{aq}{a+b}\right)$$

$$\text{and}\ s=\sqrt{\left\{\frac{2(a+b)}{\lambda ab}\right\}}\left(f-\frac{aq}{a+b}\right).$$

Let these be A_1, A_2: and after proceeding in a similar way for the sine, let the numbers in the table of $\int_s \sin\frac{\pi s^2}{2}$ corresponding to

$$s=\sqrt{\left\{\frac{2(a+b)}{\lambda ab}\right\}}\left(f+\frac{aq}{a+b}\right)$$

$$\text{and } s = \sqrt{\left\{\frac{2(a+b)}{\lambda ab}\right\}}\left(f - \frac{aq}{a+b}\right)$$

be B_1, B_2.

Then $E=$

$$\int_x \left[\sqrt{\left\{\frac{\lambda ab}{2(a+b)}\right\}}(A_1 + A_2)\cos\left\{\frac{\pi}{\lambda}\cdot\frac{a+b}{ab}\left(x - \frac{ap}{a+b}\right)^2\right\}\right.$$
$$\left. - \sqrt{\left\{\frac{\lambda ab}{2(a+b)}\right\}}(B_1 + B_2)\sin\left\{\frac{\pi}{\lambda}\cdot\frac{a+b}{ab}\left(x - \frac{ap}{a+b}\right)^2\right\}\right].$$

In the same manner $F=$

$$\int_x \left[\sqrt{\left\{\frac{\lambda ab}{2(a+b)}\right\}}(A_1 + A_2)\sin\left\{\frac{\pi}{\lambda}\cdot\frac{a+b}{ab}\left(x - \frac{ap}{a+b}\right)^2\right\}\right.$$
$$\left. + \sqrt{\left\{\frac{\lambda ab}{2(a+b)}\right\}}(B_1 + B_2)\cos\left\{\frac{\pi}{\lambda}\cdot\frac{a+b}{ab}\left(x - \frac{ap}{a+b}\right)^2\right\}\right].$$

Integrating with respect to x in exactly the same manner between $x = -e$ and $x = +e$, and putting A_3, A_4, for the numbers in the table of $\int_s \cos\frac{\pi s^2}{2}$ corresponding to

$$s = \sqrt{\left\{\frac{2(a+b)}{\lambda ab}\right\}}\left(e + \frac{ap}{a+b}\right)$$

$$\text{and } s = \sqrt{\left\{\frac{2(a+b)}{\lambda ab}\right\}}\left(e - \frac{ap}{a+b}\right),$$

and B_3, B_4 for those in the table of $\int_s \sin\frac{\pi s^2}{2}$ corresponding to the same,

$$E = \frac{\lambda ab}{2(a+b)}\left\{(A_1 + A_2)(A_3 + A_4) - (B_1 + B_2)(B_3 + B_4)\right\},$$

$$F = \frac{\lambda ab}{2(a+b)}\left\{(A_1 + A_2)(B_3 + B_4) + (B_1 + B_2)(A_3 + A_4)\right\}.$$

The intensity or $E^2 + F^2$

$$= \frac{\lambda^2 a^2 b^2}{4(a+b)^2}\left\{(A_1 + A_2)^2 + (B_1 + B_2)^2\right\}.\left\{(A_3 + A_4)^2 + (B_3 + B_4)^2\right\}$$

This expression (omitting the first factor) is the product two factors, of which one depends entirely on q and the oth depends entirely on p. If a certain value of p makes i factor small, every part of the screen for which p has th value will have a small intensity of light. A similar rema applies to the values of q which make the other factor sma Thus the screen will be crossed by bars of light of differe intensity at right angles to each other.

*75. A nearly similar investigation applies to the i vestigation of the intensity in the shadow of an opaque par lelogram with free passage for the light on all sides. It w be found that it is crossed by bars of light, the central ba being white.

76. Our limits will not allow us to examine in det these cases. The discussion of the values for particular valu of p and q depends entirely upon examination of the num rical results: and this must be done for a great number values of p and q before any conjecture can be formed as fringes, &c. about the edges. One of the simplest cases is, find the intensity of light produced by the shadow of a pla bounded by a straight line. If y is parallel to the edge, an x for the edge $= 0$, then the limits of the first integration ar

$$\text{from } y = -\infty \text{ to } y = +\infty,$$

and those of the second

$$\text{from } x = 0 \text{ to } x = \infty.$$

This case has been fully considered by M. Fresnel, and has arrived at this conclusion. If a plane be drawn throu the bright point and the edge of the plate, and if the inte section of this with the screen be called the geometric shadow: then on the dark side of the geometrical shad the intensity of the light diminishes rapidly without i creasing at all, and soon becomes insensible: but on t bright side the light increases and diminishes by sever alternations before it acquires sensibly its full brightne forming a succession of several bands on the bright side the geometrical shadow. And as, for the same point, t limits for which the tabular numbers are taken are differe

for different values of λ, and as the factor of the whole varies with λ, the intensity of the differently coloured lights will be differently proportioned at different points, and thus the bands will be coloured, nearly as in (52). This phænomenon, known by the name of *Grimaldi's coloured fringes*, had long been observed, and an imperfect explanation was given by Newton. In Fresnel's *Mémoire sur la Diffraction* it was shewn, from accurate measures with various values of a and b, to be a consequence of the theory of undulations (*Mémoires de l'Institut*, 1821).

77. Another instance is, if the form of the plate be a square *corner*, then besides the bands on the outside of the geometrical shadow there are seen within it hyperbolical curves as in fig. 21. The accurate investigation* may be performed as above: but a general explanation may be given thus. Let P and Q be points similarly situated on the two sides: the small waves diverging from their neighbourhood would, as in (48), produce bands by their interferences; and the breadth of these, by (56), would be inversely as the distance of P and Q. Consequently the nearer P and Q are taken to the solid angle, the broader will the bands be, and their form will therefore resemble the hyperbola. In this we have omitted the effects of interference of other portions of the light nearer to and further from the angle, but as the omitted parts would at different points produce effects nearly similar, it is probable that the general form of the curves will be similar to hyperbolas.

78. Another instance is, if the light fall on a very narrow slit, coloured bands of much greater breadth are thrown on the screen. The second case of (25) sufficiently explains their origin. If the slit be triangular, it is observed, (as was first remarked by Newton), that the bands are rectangular hyperbolas, the asymptotes being parallel and perpendicular to the axis of the triangle. This appears from the same investiga-

* In this and the preceding case, it is necessary to consider the effect produced by small waves diverging from distances sensibly different. In the investigation we suppose that the absolute effect of each of these is the same as the effect of a wave diverging from a surface of equal extent at a smaller distance. This is manifestly incorrect: but the inaccuracy produces no sensible error in the result, for the reason mentioned in (29).

tion, as the intervals between the bands are inversely as b the breadth of the aperture, or inversely as the distance from the geometrical shadow of the triangle's vertex.

79. In the following instance we may find the intensity at one point without much trouble. Suppose the aperture in (73) to be a round hole: to find the intensity at that point of the screen which is the projection of its center. Divide the circle into rings, the inner and outer radii of one being r and $r+\delta r$, or its surface being $2\pi r\delta r$. The distance of every point of this ring from the point of the screen is

$$b+\frac{a+b}{2ab}r^2 \text{ nearly},$$

and hence the whole displacement at the central point of the screen is

$$\textstyle\int_r 2\pi r \sin\left\{\frac{2\pi}{\lambda}\left(vt-b-\frac{a+b}{2ab}r^2\right)\right\},$$

$$\text{or } \frac{\lambda ab}{a+b}\cos\left\{\frac{2\pi}{\lambda}\left(vt-b-\frac{a+b}{2ab}r^2\right)\right\}.$$

If c be the radius of the hole, this is to be taken from $r=0$ to $r=c$, and its value is

$$\frac{2\lambda ab}{a+b}\cdot\sin\left\{\frac{2\pi}{\lambda}\left(vt-b-\frac{a+b}{4ab}c^2\right)\right\}\cdot\sin\left(\frac{2\pi}{\lambda}\cdot\frac{a+b}{4ab}c^2\right).$$

The intensity of illumination is consequently

$$\frac{4\lambda^2a^2b^2}{(a+b)^2}\cdot\sin^2\left(\frac{2\pi}{\lambda}\cdot\frac{a+b}{4ab}c^2\right).$$

On referring to (71) it will be seen that this expression is nearly similar to the expression for the intensity of the reflected light in Newton's rings, considering the denominator in (71) as constant, and making

$$V=\frac{c^2(a+b)}{2ab}:$$

and consequently the colours are nearly the same for the same values of V. To obtain the colours corresponding to

those of the inner rings, we must have V as small as possible, or $\frac{1}{a}+\frac{1}{b}$ must be as small as possible, and therefore b must be as great as possible. On diminishing b, V increases. Consequently if we first place the screen at a very great distance and then bring it nearer to the aperture, the series of colours at the center will be the same as those found on tracing Newton's rings outwards: but as we cannot make $\frac{1}{a}+\frac{1}{b}=0$, we cannot have all the orders beginning from the central black. This agrees with observation. For any other point of the screen, the intensity can be found only by the general method of (73).

*79. Instead of supposing a circular hole in a plate of unlimited extent, suppose the plate to be a small circular disk, with free passage for the light on all sides. The whole displacement will be expressed by the same formula

$$\textstyle\int_r 2\pi r \sin\left\{\frac{2\pi}{\lambda}\left(vt-b-\frac{a+b}{2ab}\,r^2\right)\right\};$$

but the limits of r will now be from c to ∞.

Let $\frac{\pi(a+b)}{\lambda ab}\,r^2=s$: the expression becomes

$$\frac{\lambda ab}{a+b}\textstyle\int_s \sin\left\{\frac{2\pi}{\lambda}(vt-b)-s\right\};$$

and the limits of s are $\frac{\pi(a+b)}{\lambda ab}\,c^2$ and ∞. If we perform the integration in this state, we come upon the unintelligible symbol $\cos\infty$. To avoid this, and at the same time to represent an intensity of little waves slowly decreasing as the inclination increases, use the expression

$$\frac{\lambda ab}{a+b}\textstyle\int_s \epsilon^{-fs}\sin\left\{\frac{2\pi}{\lambda}(vt-b)-s\right\},$$

where f may be so small that its effect through a considerable angle is almost insensible. The general integral is

$$\frac{\lambda ab}{(a+b)(1+f^2)}\,\epsilon^{-fs}\left[\cos\left\{\frac{2\pi}{\lambda}(vt-b)-s\right\}-f\sin\left\{\frac{2\pi}{\lambda}(vt-b)-s\right\}\right];$$

which vanishes for $s=\infty$, however small f may be. Omitting the trifling effect of f in the first value of s, the limited integral becomes

$$-\frac{\lambda ab}{a+b}\cos\left\{\frac{2\pi}{\lambda}\left(vt-b-\frac{a+b}{2ab}c^2\right)\right\};$$

and the intensity of light is $\frac{\lambda^2a^2b^2}{(a+b)^2}$, in which the symbol c does not appear. The intensity of light, therefore, at the center, is the same as if $c=0$; that is, the same as if there were no disk in the way.

For the intensity of light at any excentric point, the reader is referred to the *Philosophical Magazine*, 1841, January.

PROP. 20. Every thing remaining as in the last problem, except that, close to the hole, a lens is placed of such focal length that light diverging from A will be made to converge to O: to find the intensity of light on the screen.

80. From (44) it appears that the form of the front of the wave after refraction by the lens will be a sphere of which O is the center. Let O, fig. 22, be the origin of co-ordinates: p and q the co-ordinates of a point M on the screen: x, y, z, those of P, z being parallel to OA. Then

$$PM^2=(p-x)^2+(q-y)^2+z^2.$$

But by the equation to the surface of a sphere,

$$x^2+y^2+z^2=b^2:$$

hence

$$PM^2=b^2+p^2+q^2-2px-2qy,$$

$$\text{and } PM=b+\frac{p^2+q^2}{2b}-\frac{px}{b}-\frac{qy}{b} \text{ nearly.}$$

The terms depending on x^2 and y^2 will be insensible, as they will be multiplied by the very small quantities p^2 and q^2. Put B for $b+\frac{p^2+q^2}{2b}$: then, as in the first and last parts of (73), the whole displacement at M is

$$\int_x\int_y \sin\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{px}{b}+\frac{qy}{b}\right)\right\}.$$

This expression is much simpler than that of (73), as there are no terms involving x^2 and y^2. The first integration can always be performed: it gives

$$-\frac{b\lambda}{2\pi q}\cos\left\{\frac{2\pi}{\lambda}\left(vt - B + \frac{px}{b} + \frac{qy}{b}\right)\right\};$$

and if y' and y'' are the least and greatest values of y for a given value of x (given by the equation to the aperture in terms of x), the first integral is, between these limits,

$$\frac{b\lambda}{2\pi q}\left[\cos\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{px}{b}+\frac{qy'}{b}\right)\right\}-\cos\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{px}{b}+\frac{qy''}{b}\right)\right\}\right]$$

$$=\cos\left\{\frac{2\pi}{\lambda}(vt-B)\right\}\frac{b\lambda}{2\pi q}\left[\cos\left\{\frac{2\pi}{b\lambda}(px+qy')\right\}-\cos\left\{\frac{2\pi}{b\lambda}(px+qy'')\right\}\right]$$

$$-\sin\left\{\frac{2\pi}{\lambda}(vt-B)\right\}\frac{b\lambda}{2\pi q}\left[\sin\left\{\frac{2\pi}{b\lambda}(px+qy')\right\}-\sin\left\{\frac{2\pi}{b\lambda}(px+qy'')\right\}\right].$$

Let the integrals of the terms within the brackets (with respect to x and between the proper limits) be P and Q: then the coefficients of

$$\cos\left\{\frac{2\pi}{\lambda}(vt-B)\right\} \text{ and } \sin\left\{\frac{2\pi}{\lambda}(vt-B)\right\},$$

$$\text{are respectively } \frac{b\lambda}{2\pi q}P \text{ and } -\frac{b\lambda}{2\pi q}Q,$$

and the intensity of the light is $\dfrac{b^2\lambda^2}{4\pi^2 q^2}(P^2+Q^2)$.

81. Ex. Let the aperture be a parallelogram whose sides are $2e$ and $2f$ in the direction of x and y. Here

$$y' = -f,\quad y'' = +f:$$

$$\cos\left\{\frac{2\pi}{b\lambda}(px+qy')\right\} - \cos\left\{\frac{2\pi}{b\lambda}(px+qy'')\right\}$$

$$= \cos\left\{\frac{2\pi}{b\lambda}(px-qf)\right\} - \cos\left\{\frac{2\pi}{b\lambda}(px+qf)\right\}$$

$$= 2\sin\left(\frac{2\pi}{b\lambda}px\right).\sin\left(\frac{2\pi}{b\lambda}qf\right),$$

the integral of which is

$$-\frac{b\lambda}{\pi p}\sin\left(\frac{2\pi}{b\lambda}qf\right).\cos\left(\frac{2\pi}{b\lambda}px\right);$$

which from $x=-e$ to $x=+e$ gives $P=0$. Also

$$\sin\left\{\frac{2\pi}{b\lambda}(px+qy')\right\}-\sin\left\{\frac{2\pi}{b\lambda}(px+qy'')\right\}$$

$$=\sin\left\{\frac{2\pi}{b\lambda}(px-qf)\right\}-\sin\left\{\frac{2\pi}{b\lambda}(px+qf)\right\}$$

$$=-2\cos\frac{2\pi px}{b\lambda}.\sin\frac{2\pi qf}{b\lambda},$$

the integral of which is

$$-\frac{b\lambda}{\pi p}\sin\frac{2\pi qf}{b\lambda}.\sin\frac{2\pi px}{b\lambda},$$

which from $x=-e$ to $x=+e$ gives

$$Q=-\frac{2b\lambda}{\pi p}\sin\frac{2\pi qf}{b\lambda}.\sin\frac{2\pi pe}{b\lambda}.$$

Hence the intensity is

$$\frac{b^4\lambda^4}{\pi^4 p^2q^2}\sin^2\frac{2\pi qf}{b\lambda}.\sin^2\frac{2\pi pe}{b\lambda},$$

$$\text{or } 16e^2f^2\left(\frac{b\lambda}{2\pi qf}\sin\frac{2\pi qf}{b\lambda}\right)^2\left(\frac{b\lambda}{2\pi pe}\sin\frac{2\pi pe}{b\lambda}\right)^2.$$

This expression is maximum when $p=0$, $q=0$: so that there is a bright point in the place of the image determined by common Optics. It is 0 when p is any multiple of $\frac{b\lambda}{2e}$, or when q is any multiple of $\frac{b\lambda}{2f}$. This shews that the screen is traversed by rectangular dark bars at equal intervals, the intervals in the direction of the length of the parallelogram being shorter than the others. For a given value of p, the brightness is greatest when $q=0$, or when q has one of the values

which makes $\frac{b\lambda}{2\pi qf} \sin \frac{2\pi qf}{b\lambda}$ maximum. Thus it appears that there will be a brilliant point at the center; a four-rayed cross through the center, the rays being interrupted at intervals; and a series of less bright patches in square arrangement in the angles of the cross: also the distances from the center are greater for the red rays than for the blue. When the parallelogram is narrow, the bright parts in the direction of one side form one of the kinds of spectra described by Fraunhofer.

82. Let the aperture be an equilateral triangle. Take x in the direction of the perpendicular to one side, and let the angle opposite this side be the origin of co-ordinates: let e be the whole length of the perpendicular. Then

$$y' = -x \tan 30^0 : \quad y'' = +x \tan 30^0.$$

Hence

$$P = \int_x \left[\cos\left\{\frac{2\pi x}{b\lambda}(p - q\tan 30^0)\right\} - \cos\left\{\frac{2\pi x}{b\lambda}(p + q\tan 30^0)\right\}\right]$$

$$= \frac{b\lambda}{2\pi(p - q\tan 30^0)} \sin\left\{\frac{2\pi x}{b\lambda}(p - q\tan 30^0)\right\}$$

$$- \frac{b\lambda}{2\pi(p + q\tan 30^0)} \sin\left\{\frac{2\pi x}{b\lambda}(p + q\tan 30^0)\right\},$$

the value of which from $x = 0$ to $x = e$ is found by putting e for x. And

$$Q = \int_x \left[\sin\left\{\frac{2\pi x}{b\lambda}(p - q\tan 30^0)\right\} - \sin\left\{\frac{2\pi x}{b\lambda}(p + q\tan 30^0)\right\}\right],$$

the value of which from $x = 0$ to $x = e$ is

$$\frac{b\lambda}{2\pi(p - q\tan 30^0)}\left[1 - \cos\left\{\frac{2\pi e}{b\lambda}(p - q\tan 30^0)\right\}\right]$$

$$- \frac{b\lambda}{2\pi(p + q\tan 30^0)}\left[1 - \cos\left\{\frac{2\pi e}{b\lambda}(p + q\tan 30^0)\right\}\right].$$

The sum of the squares is $\left(\text{omitting the factor } \frac{b^2\lambda^2}{4\pi^2}\right)$

$$\frac{1}{(p-q\tan 30^0)^2}\left[2-2\cos\left\{\frac{2\pi e}{b\lambda}(p-q\tan 30^0)\right\}\right]$$

$$+\frac{1}{(p+q\tan 30^0)^2}\left[2-2\cos\left\{\frac{2\pi e}{b\lambda}(p+q\tan 30^0)\right\}\right]$$

$$-\frac{2}{p^2-q^2\tan^2 30^0}\times$$

$$\left[1+\cos\frac{4\pi eq\tan 30^0}{b\lambda}-\cos\left\{\frac{2\pi e}{b\lambda}(p-q\tan 30^0)\right\}-\cos\left\{\frac{2\pi e}{b\lambda}(p+q\tan 30^0)\right\}\right]$$

$$=\frac{2p^2+6q^2\tan^2 30^0}{(p^2-q^2\tan^2 30^0)^2}-\frac{4pq\tan 30^0+4q^2\tan^2 30^0}{(p^2-q^2\tan^2 30^0)^2}\cos\left\{\frac{2\pi e}{b\lambda}(p-q\tan 30^0)\right\}$$

$$+\frac{4pq\tan 30^0-4q^2\tan^2 30^0}{(p^2-q^2\tan^2 30^0)^2}\cos\left\{\frac{2\pi e}{b\lambda}(p+q\tan 30^0)\right\}$$

$$-\frac{2p^2-2q^2\tan^2 30^0}{(p^2-q^2\tan^2 30^0)^2}\cos\frac{4\pi eq\tan 30^0}{b\lambda}.$$

Let $p=r\cos\theta$, $q=r\sin\theta$: which is the same as referring M to the central point of the screen by polar co-ordinates. Then observing that $\tan^2 30=\frac{1}{3}$, and restoring the factors

$$\frac{b^2\lambda^2}{4\pi^2} \text{ and } \frac{b^2\lambda^2}{4\pi^2q^2},$$

this may be put in the form

$$\frac{3b^4\lambda^4}{32\pi^4r^4}\cdot\frac{1}{\sin^2\theta\,.\sin^2(\theta-60^0)\,.\sin^2(\theta-120^0)}$$

$$\times\left[\frac{3}{4}-\sin(\theta-60^0)\,.\sin(\theta-120^0)\,.\cos\left(\frac{4\pi re}{b\lambda\sqrt{(3)}}\sin\theta\right)\right.$$

$$-\sin(\theta-120^0)\,.\sin(\theta-180^0)\,.\cos\left\{\frac{4\pi re}{b\lambda\sqrt{(3)}}\sin(\theta-60^0)\right\}$$

$$\left.-\sin(\theta-180^0)\,.\sin(\theta-240^0)\,.\cos\left\{\frac{4\pi re}{b\lambda\sqrt{(3)}}\sin(\theta-120^0)\right\}\right].$$

The maximum value it will be found is when $r=0$, and is $=\frac{e^4}{3}$. The value is also considerable when $\theta=0$, or $=60^0$, or $=120^0$, or $=180^0$, or $=240^0$, or $=300^0$, when it is

$$\frac{\lambda^2e^2b^2}{3\pi^2r^2}\left(1-\frac{\lambda b}{\pi re}\sin\frac{2\pi re}{\lambda b}\right)+\frac{\lambda^4b^4}{6\pi^4r^4}\left(1-\cos\frac{2\pi re}{\lambda b}\right),$$

(as will be found by commencing with the first integral formula of (80), and, for the ray, making $q=0$; and, for the central point, making p also $=0$). This points out exactly the star-like form observed by Sir J. Herschel (*Encycl. Metrop.* Light, Art. 772).

83. Let the aperture be a great number m of equal parallelograms of length $2f$ and breadth e at equal distances g. Here $y'=-f$, $y''=+f$: and the expression to be integrated is

$$\cos\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{p}{b}x-\frac{qf}{b}\right)\right\}-\cos\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{p}{b}x+\frac{qf}{b}\right)\right\}$$

$$=2\sin\frac{2\pi qf}{\lambda b}.\sin\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{p}{b}x\right)\right\}.$$

The general integral is

$$-\frac{\lambda b}{\pi p}\sin\frac{2\pi qf}{\lambda b}.\cos\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{p}{b}x\right)\right\}.$$

If k be the value of x corresponding to the first side of the first parallelogram, that corresponding to the first side of the $(n+1)^{\text{th}}$ parallelogram will be $k+n(e+g)$, and that corresponding to its last side $k+n(e+g)+e$. The integral therefore for the $(n+1)^{\text{th}}$ parallelogram is

$$\frac{\lambda b}{\pi p}\sin\frac{2\pi qf}{\lambda b}.\left[\cos\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{pk}{b}+\frac{pn(e+g)}{b}\right)\right\}\right.$$

$$\left.-\cos\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{pk}{b}+\frac{pn(e+g)}{b}+\frac{pe}{b}\right)\right\}\right]$$

$$=\frac{2\lambda b}{\pi p}\sin\frac{2\pi qf}{\lambda b}.\sin\frac{\pi pe}{\lambda b}.\sin\left[\frac{2\pi}{\lambda}\left\{vt-B+\frac{pk}{b}+\frac{pe}{2b}+\frac{p(e+g)}{b}n\right\}\right].$$

Let $B-\frac{pk}{b}-\frac{pe}{2b}=C$: then the whole displacement at M produced by all the parallelograms, {restoring the factor $\frac{b\lambda}{2\pi q}$ from (80)}, is

$$\frac{\lambda^2 b^2}{\pi^2 pq}\sin\frac{2\pi qf}{\lambda b}\cdot\sin\frac{\pi pe}{\lambda b}\Sigma\sin\left[\frac{2\pi}{\lambda}\left\{vt-C+\frac{p(e+g)}{b}n\right\}\right].$$

The finite integral of the last term with respect to n is

$$\frac{-1}{2\sin\frac{p(e+g)\pi}{b\lambda}}\cos\left[\frac{2\pi}{\lambda}\left\{vt-C+\frac{p(e+g)}{b}\,(n-\tfrac{1}{2})\right\}\right],$$

which from $n=0$ to $n=m$ is

$$\frac{\sin\frac{mp(e+g)\pi}{b\lambda}}{\sin\frac{p(e+g)\pi}{b\lambda}}\sin\left[\frac{2\pi}{\lambda}\left\{vt-C+\frac{p(e+g)}{b}\cdot\frac{m-1}{2}\right\}\right].$$

Thus, putting

$$C-\frac{p(e+g)}{b}\cdot\frac{m-1}{2}=D,$$

we have for the whole displacement

$$\frac{\lambda^2 b^2}{\pi^2 pq}\sin\frac{2\pi qf}{\lambda b}\cdot\sin\frac{\pi pe}{\lambda b}\cdot\frac{\sin\frac{mp(e+g)\pi}{b\lambda}}{\sin\frac{p(e+g)\pi}{b\lambda}}\cdot\sin\left\{\frac{2\pi}{\lambda}(vt-D)\right\}:$$

and consequently the intensity of the light is

$$4e^2f^2\left(\frac{\lambda b}{2\pi qf}\cdot\sin\frac{2\pi qf}{\lambda b}\right)^2\left(\frac{\lambda b}{\pi pe}\cdot\sin\frac{\pi pe}{\lambda b}\right)^2\left\{\frac{\sin\frac{p(e+g)\pi}{b\lambda}m}{\sin\frac{p(e+g)\pi}{b\lambda}}\right\}^2.$$

84. The most remarkable variation of this depends on the last term. This may be represented by $\left(\frac{\sin m\theta}{\sin\theta}\right)^2$, where

m is a large whole number. This has a great number of maxima corresponding to values of θ for which $m\theta$ is somewhat less than the successive odd multiples of $\frac{\pi}{2}$ (omitting $\frac{\pi}{2}$ itself); but the greatest maximum is that found by making $\sin\theta = 0$. Its value is then m^2, which is much greater than any of the others. For, the next maximum, when $m\theta = 258^0$ nearly, is

$$\left(\frac{\sin 258^0}{\sin \frac{258^0}{m}}\right)^2 \text{nearly} = m^2 \times 0{\cdot}0472\,;$$

the next is nearly $= m^2 \times 0{\cdot}016$: &c.; and when $\sin\theta$ is nearly $= 1$, the maximum is nearly 1. As we approach to the value $\theta = \pi$, one or two values are sensible, and then we reach the large maximum, of the same value as before. Suppose now we have placed on the object-glass of a telescope a grating consisting of 100 parallel wires. There will be a bright image of the luminous point formed at the centre of the field, and one or two less bright on each side, so close that they cannot be distinguished: after this there will be others, but their intensity will diminish so rapidly (being at one of the *maxima* only $\frac{1}{10000}$ of that of the brightest) that they will be invisible; and at a distance there will be another point as bright as the first: and at an equal distance beyond it, another: and so on. Thus there will be a succession of luminous points at equal distances from each other, with no perceptible light between them. The distance of these points is found by making

$$\theta = 0, \text{ or } \pi, \text{ or } 2\pi \text{ \&c.; or } p = 0, \text{ or } \frac{b\lambda}{e+g}, \text{ or } \frac{2b\lambda}{e+g}, \text{ \&c.}$$

This applies to any one kind of homogeneous light. When there is a mixture of differently coloured lights (as in white light), there will be a union of bright points of all the colours where $p = 0$, but at no other place. For, to arrive at the second bright point, we must go to a distance from the first proportional to λ. Consequently the next blue point will be

nearer to the center than the next red point, &c. Thus in the center there will be a bright white point, but at the sides there will be spectra similar to those formed by a prism, their blue ends being nearest to the center. And as each bright point is almost perfectly insulated, the spectrum will be *pure;* that is, there will be no sensible mixture of colours in any part of it. This is verified completely by experiment: the spectra are so pure that, when the solar light is used, the fixed lines or interruptions of the spectrum, which are so delicate that only the best prisms will shew them, may be seen in the spectra formed as we have described.

85. We shall now consider the term

$$\left(\frac{\lambda b}{\pi p e} \cdot \sin \frac{\pi p e}{\lambda b}\right)^2.$$

When p is small, or when e is small, this is $= 1$. When p is increased to any multiple of $\frac{\lambda b}{e}$, it vanishes. Consequently, whenever the same value of p is a multiple of $\frac{\lambda b}{e}$ and of $\frac{\lambda b}{e+g}$, one of the spectra will disappear: that is, whenever e and g are commensurate. This is true in experiment. And at all events, the successive spectra are less bright than the central colourless image, this factor having its greatest value when $p = 0$.

It is unnecessary to consider the effect of the first factor, as it only points out the law of brightness in the direction of the length of the parallelograms.

86. The aperture is a circle, whose radius $= e$. (This is the ordinary state of a telescope.) Since the intensity will be equal at equal distances in all directions from the center of the screen, let $p = r$, $q = 0$. It is evident that

$$y' = -\sqrt{(e^2 - x^2)}, \quad y'' = +\sqrt{(e^2 - x^2)}.$$

Then the expression to be integrated in (80) becomes

$$\int_x \int_y \sin\left\{\frac{2\pi}{\lambda}\left(vt - B + \frac{rx}{b}\right)\right\}.$$

The first integral is obtained by multiplying the sine by y. Taking this between the limits, the expression becomes

$$\int_x 2\sqrt{(e^2-x^2)}\sin\left\{\frac{2\pi}{\lambda}\left(vt-B+\frac{rx}{b}\right)\right\},$$

or

$$2\sin\left\{\frac{2\pi}{\lambda}(vt-B)\right\}\int_x\sqrt{(e^2-x^2)}\cos\frac{2\pi rx}{b\lambda}$$
$$+2\cos\left\{\frac{2\pi}{\lambda}(vt-B)\right\}\int_x\sqrt{(e^2-x^2)}\sin\frac{2\pi rx}{b\lambda},$$

which is to be integrated from $x=-e$ to $x=+e$.

The second line, it is evident, will vanish: and the expression is reduced to

$$2\sin\left\{\frac{2\pi}{\lambda}(vt-B)\right\}\int_x\sqrt{(e^2-x^2)}\cos\frac{2\pi rx}{b\lambda}.$$

Let $\dfrac{x}{e}=w$, $\dfrac{2\pi re}{b\lambda}=n$: the expression becomes

$$2e^2\sin\left\{\frac{2\pi}{\lambda}(vt-B)\right\}\int_w\sqrt{(1-w^2)}\cos nw,$$

from $w=-1$ to $w=+1$.

$$\text{Let } \phi(n)=\frac{4}{\pi}\int_w\sqrt{(1-w^2)}\cos nw,$$

from $w=0$ to $w=1$:

the expression is

$$\sin\left\{\frac{2\pi}{\lambda}(vt-B)\right\}\times e^2\pi\,.\,\phi(n);$$

and the intensity of light is

$$e^4\pi^2\{\phi(n)\}^2.$$

A table of values of $\phi(n)$ is given at the end of this work. On examining it, it will be seen that there is blackness when $n=3{,}83$ or $7{,}14$ or $10{,}17$ (which indicates dark rings): and brightness when $n=5{,}12$ or $8{,}43$ or $11{,}63$ (which indicates

bright rings), with intensities respectively about $\frac{1}{57}$, $\frac{1}{240}$, and $\frac{1}{620}$, of that at the center.

The angular diameter of any ring as viewed from the center of the object-glass $= \frac{r}{b} = \frac{\lambda n}{2\pi e}$. Let this angular diameter in seconds be s, then $n = \frac{2\pi es}{\lambda} \sin 1''$. If e the radius of the object-glass be expressed in inches, and if for mean rays λ be 0,000022 inch, the last equation becomes

$$n = 1{,}3846 \times es.$$

Hence we have,

For the bright rings, $es = 3{,}70,\ 6{,}09,\ 8{,}40$;

For the black rings, $es = 2{,}76,\ 5{,}16,\ 7{,}32$.

These are the ordinary rings seen round a star when viewed with a good telescope.

It appears from this that the central image of a star (the star being considered as a point of light without visible dimensions) is not a point but a circular disk, diminishing in intensity of light towards the edges; whose extreme radius, in seconds, is defined by the equation

$$es = 2{,}76.$$

The larger is the aperture of the telescope, the smaller is the angular measure of the breadth of the disk.

*86. The aperture is an ellipse, whose semiaxes are e and f. Taking the general co-ordinates p and q for a point in the screen, and remarking that, in the expressions of (80),

$$y' = -\frac{f}{e}\sqrt{(e^2 - x^2)},$$

$$\text{and } y'' = +\frac{f}{e}\sqrt{(e^2 - x^2)},$$

it will be seen that the first of the four terms in the final expression for displacement is

$$\cos\left\{\frac{2\pi}{\lambda}(vt-B)\right\}\frac{b\lambda}{2\pi q}\cos\left[\frac{2\pi}{b\lambda}\left\{px-\frac{fq}{e}\sqrt{(e^2-x^2)}\right\}\right],$$

to be integrated from $x=-e$ to $x=+e$.

Now if we had investigated the displacement with a circular aperture of radius e, for a point on the screen whose co-ordinates are p and q', the first of the four terms would have been

$$\cos\left\{\frac{2\pi}{\lambda}(vt-B)\right\}\frac{b\lambda}{2\pi q'}\cos\left[\frac{2\pi}{b\lambda}\{px-q'\sqrt{(e^2-x^2)}\}\right].$$

Disregarding the change in the constant multiplier from $\frac{b\lambda}{2\pi q}$ to $\frac{b\lambda}{2\pi q'}$, these expressions would be exactly the same and to be integrated between the same limits, if $\frac{fq}{e}=q'$: and the same holds for the other terms of the displacement. But it was found in (86) that, for the circular aperture, the brightness or darkness depends simply upon the value of r, where $r^2=p^2+q'^2$: and a ring of definite light is determined by the equation

$$r^2=\text{constant, or } p^2+q'^2=\text{constant}.$$

Therefore in the case of the elliptic aperture, a ring of definite light is determined by the equation

$$p^2+q'^2=\text{constant, or } p^2+\frac{f^2q^2}{e^2}=\text{constant}.$$

This is the equation to an ellipse whose semiaxis in the direction of p is to that in the direction of q as f to e. Consequently the central spot and the rings are elliptical, but the direction of the major axis of the rings corresponds to that of the minor axis of the aperture. This is found in experiment to be true.

87. The whole of the experiments which are the subject of Prop. 20, (80) to (*86), are easily made by limiting the aperture of the object-glass of a telescope, or by placing

gratings before it. The appearances which we have investigated are those that would be formed on a screen in the focus of the object-glass; but it is well known by common Optics that the appearance presented to the eye, when an eye-glass is applied whose focus coincides with the focus of the object-glass, is just the same as if the light had been received on a screen placed there. Thus it is only necessary to limit the aperture and then to view a bright point (as a star), when the phænomena that we have described will be seen in great perfection.

88. The experiment of (83) &c. is particularly remarkable on this account. It shews that there is light diverging in all directions from the front of the grand wave where it meets the lens, which is insensible only because it is destroyed by other light. For if we view a luminous point with a telescope in its usual state, no side images are seen: on putting a grating on the object-glass, which *intercepts* a part of the light, the side images are visible. That this depends simply on the obstruction of the light, and not on any reflection or refraction by the grating, is evident from this circumstance, that it is indifferent whether the grating consist of wire, or silk, or lines scratched on the glass with a diamond point, or lines ruled on a film of soap or grease. The same principle may be used to explain the spectra produced by the reflection of light from metallic surfaces on which lines are engraved at very small equal distances. In fig. 23 if light from F falls on a small reflecting surface Ad and is received on a screen GH, and if F and G be both distant, then a point G may be found such that the paths FAG, FBG, &c. will not sensibly differ in length; and therefore the small waves which are produced by the same great wave, coming from every part of the surface, will meet in the same phase at G. And this will be true whether any part of the surface is removed or not. But at H there will be no illumination, because we may divide the surface into parts A, a, B, b, &c. such that the path FaH (supposing the surface a continuous plane) is less than FAH by $\frac{\lambda}{2}$, and therefore the small wave coming from a will destroy that coming from A; the small wave coming from b will destroy that coming from B: and so on. Now suppose that

we remove the parts a, b, c, d, &c. There is now no wave to destroy any one of those coming from A, B, &c.: and they will not destroy each other, because the path FBH being less than FAH by λ, FCH being less than FBH by λ, &c., they are all in the same phase. Consequently there will be brightness at H. For different values of λ it is evident that we must take points at different distances from G: and thus spectra will be formed nearly as in (84).

For calculations applying to various cases of interference, the reader is referred to several volumes of the *Philosophical Transactions*, the *Cambridge Transactions*, and the *Philosophical Magazine*. A most remarkable investigation by Professor Stokes on an apparent change in the periodic time of waves will be found in the *Philosophical Transactions*, 1852.

APPLICATION OF THE THEORY OF UNDULATIONS TO THE PHÆNOMENA OF POLARIZED LIGHT.

89. In the preceding investigations, no reference has been made to the direction in which the particles of the luminiferous ether vibrate. They might, like the particles of air in the transmission of sound, vibrate in the direction in which the wave is passing: or they might, like the particles of a musical string, vibrate perpendicularly to the direction of the wave, but all in one plane passing through that direction. To these, or any other conceivable vibrations, our investigations would apply equally well: all that is required being that they should be subject to the general law of undulations, and that for a considerable number of vibrations the extent and time of vibration should be the same. The phænomena of polarization, however, enable us to point out what is the kind of vibration.

90. The properties of Iceland spar (which, it has since been discovered, are possessed by the greater number of transparent crystals) first pointed out the characteristic law of polarization. If a pencil of common light be made to pass through a rhombohedron of this crystal, it is separated into two of equal intensity. This may be shewn either by viewing a small object through it, when two images will

be seen; or by placing it behind a lens on which the light of the Sun or that of a lamp is thrown, when two images will be formed at the focus. A line drawn through those two images is in the direction of the shorter diagonal of the rhombic face of the crystal; the rhombohedron being supposed to be bounded by planes of cleavage, and the pencil of light being incident perpendicular to one of them.

91. On examining the paths of these pencils within the crystal, it is found that one of them obeys the ordinary laws of refraction, but the other follows a more complicated law (which we shall hereafter describe). The first is therefore called the Ordinary pencil, and the other the Extraordinary pencil: and they are frequently denoted by the letters O and E.

92. To the eye no difference is discoverable between the two pencils, or between either of them and a pencil of common light whose intensity is the same. Yet the properties of the light in the two pencils are different, and both are different from common light. For if we take one of the pencils (for instance O) and place a second rhombohedron before it; on turning the first rhombohedron it is found that in general the second crystal separates the pencil O into two of *unequal* intensity, one following the ordinary law and the other the extraordinary law (which we may call Oo and Oe), and that in certain relative positions of the crystal one of the pencils wholly disappears. On examining the positions it is found that, when the two rhombohedrons are in similar positions (that is, when all the planes of cleavage of one are parallel to those of the other), or when they are in opposite positions (that is when, keeping the same surfaces in contact, the first is turned 180° from the position just described), Oe disappears, and Oo only remains; that is, there is only an ordinary pencil produced by the second crystal. On the contrary, when the first rhombohedron is turned 90° either way from the position first described, Oo disappears, and Oe only remains: that is, there is only an extraordinary pencil produced. In any intermediate position that pencil is strongest which, in the nearest of the four positions that we have mentioned, does not vanish.

93. Now if instead of O we take the pencil E, the appearances are wholly changed. In general, as before, the second rhombohedron divides this into two pencils of unequal intensity, one following the ordinary and the other the extraordinary law (which we shall call Eo and Ee). But when the crystals are in similar or in opposite positions, Eo vanishes, and Ee only remains: that is, there is only an extraordinary pencil produced. When one is turned 90^0 from the similar position, Ee vanishes and Eo remains: that is, there is only an ordinary pencil produced.

94. It appears then that neither* of these two pencils is similar to common light; for, when either of them is received on a second rhombohedron, it does not always produce two pencils: common light always does produce two. It appears also that they are not similar to each other; for, in certain positions of the second rhomb, O will produce only an ordinary ray, while E will produce only an extraordinary ray: in certain other positions, O will produce only an extraordinary ray, and E only an ordinary ray. The rays therefore have some peculiar properties depending on the position of the crystal. But between the properties of the two rays a remarkable relation can be found. When the rhombohedrons are in similar positions, O will produce only an ordinary ray. When the first is turned 90^0, E will produce only an ordinary ray. Consequently, on turning the crystal 90^0, E has the same properties which O had before turning it. Again, when the rhombohedrons are in similar positions, E will produce only an extraordinary ray. On turning the first through 90^0, O will produce only an extraordinary ray. Consequently, on turning the crystal 90^0, O has the same properties which E had before turning it. This shews clearly that the two pencils have properties of the same kind with reference to two planes passing through their direction and moving with the crystal; and that the two planes are at right angles to each other. If a plane passing through the direction of the pencil and the shorter diagonal of the rhombic face be called the

* The reader will observe that the term Ordinary pencil does not signify that the pencil is similar in its properties to common light, but merely that it is subject to the same laws of refraction as common light.

principal plane* of the crystal, then we may assert that the properties of the Ordinary ray possess the same relation to the principal plane, which the properties of the Extraordinary ray possess to the plane at right angles to the principal plane. This is commonly expressed thus: the Ordinary ray is *polarized in the principal plane*, and the Extraordinary ray is *polarized in a plane perpendicular to the principal plane.*

95. There are some crystals which possess the property of separating common light into an Ordinary and an Extraordinary pencil, and then absorbing one of them. Thus certain specimens of agate, and plates of tourmaline cut parallel to the axis, allow the Extraordinary pencil to pass, and nearly suppress the Ordinary. This however is only true when the plates have a certain thickness: for, when they are very thin, the Ordinary and Extraordinary pencils are seen to have equal intensities. But the method of exhibiting polarized light which is most extensively used in experiments is, to reflect common light from unsilvered glass or any other transparent substance, solid or fluid. It is found that if the tangent of the angle of incidence is equal to the refractive index, the whole of the reflected light is *polarized*† in the same way as the Ordinary ray produced by the first rhombohedron of Iceland spar when its principal plane is parallel to the plane of reflection from the unsilvered glass, &c. For, if the second rhombohedron be placed in that position to receive the reflected ray (instead of receiving the ray from the first rhombohedron), an ordinary ray only is produced: if in the position differing 90° from this, an extraordinary ray only is produced: which (92) is exactly the same effect as that produced by O, the crystal having the position that we have described. This is expressed by saying that the reflected light is *polarized in the plane of reflection.* The angle of incidence which is proper for this is called the *polarizing angle.* The transmitted light, it is found, possesses in part the properties of the Extraordinary ray; the principal plane of the crystal

* This term will be used hereafter in a more general sense.

† This was the discovery of Malus. It was important at the time, as calling the attention of philosophers to the subject: but all the phænomena of coloured rings, &c. may be exhibited perfectly well without using this property of reflection.

with which we mentally compare it being still conceived to be parallel to the plane of reflection. For, if the second rhomb be placed in that position, the transmitted light produces both an ordinary and an extraordinary ray, but the former is less bright than the latter. This is expressed by saying that the transmitted light is *partially polarized in the plane perpendicular to the plane of reflection.* If a great number of plates of unsilvered glass be placed together, the reflected light appears completely polarized in the plane of reflection, and the transmitted light appears completely polarized in the plane perpendicular to the plane of reflection.

96. We have here considered the test of polarization to be, the formation of only one ray when the light passes through a crystal of Iceland spar. But as the reflection from unsilvered glass at the polarizing angle gives properties exactly similar to those of the Ordinary ray of Iceland spar (the principal plane of the spar being conceived parallel to the plane of reflection), it may be conjectured that light polarized in the plane perpendicular to the plane of reflection, as it would not furnish any Ordinary ray with Iceland spar, will not furnish a reflected ray from unsilvered glass, but will be entirely transmitted. This is verified by experiment: and thus we have an easy practical test of the polarization of light. If light incident at the polarizing angle on unsilvered glass is not susceptible of reflection, it is polarized in the plane perpendicular to the plane of reflection. And if, on turning the glass round the incident ray without varying the inclination, the reflected light does not vanish in any position of the glass, the light is not polarized. In the same manner the polarization of a ray may be ascertained by examining the state of the emergent ray, after incidence on a plate of tourmaline; if in any position of the tourmaline the emergent ray disappears, the plane of polarization of the incident ray is parallel to the plane then passing through the ray and through the axis of the tourmaline.

97. From this it will easily be inferred that if two such plates of tourmaline are placed with their axes at right angles to each other, no light can pass through them. For the light which is transmitted by the first is polarized in the plane per-

pendicular to its axis, that is, in the plane of the axis of the second: and therefore is not allowed to pass through the second. If one of the tourmalines be turned, light is immediately seen: it increases till the axes of the tourmalines are parallel. In the same manner if in fig. 24 A be a plate or several parallel plates of unsilvered glass, and B an unsilvered glass* whose posterior surface is blackened, to prevent reflection there, B being fixed on a block which turns round a spindle at C in the direction of AB: and if each of the glass surfaces make with AB an angle of about $33^\circ.13'$ (the refractive index of plate glass for mean rays being about $1.527 = \tan 56^\circ.47'$): then on receiving the light of the clouds on A in such a direction that the reflected light falls on B, and placing the eye to receive the light reflected from B, it is seen that when the planes of reflection coincide, or nearly coincide, a considerable quantity of light is reflected from B: as the planes of reflection are inclined, less light is reflected: and when (as in the figure) they are perpendicular to each other, no light is reflected. This is strictly true only for the light incident from A on B in direction of the line joining their centers: but it is nearly true for light making a small angle with this. It is strictly true also for light of only one colour (since the polarizing angle, which depends on the refractive index, is different for differently coloured rays), but if the mean angle be used it is nearly true for all. We shall frequently allude to this apparatus: we shall then call A the *polarizing plate* and B the *analyzing plate.*

98. Now in the experiment of (46) or (57), (fig. 14 and 15) if a plate of tourmaline be placed in each of the pencils of light supposed to start from G and H, the plates of tourmaline being cut from the same large plate which has been carefully worked to uniform thickness, it is found that the existence of the fringes of interference depends entirely on the relative position of the axes of the tourmaline plates. If both axes be

* Instead of reflection from an unsilvered glass, transmission through a plate of tourmaline may be used. Still more convenient than a plate of tourmaline is the "Nicol's Prism," a combination of two prisms cut out of a block of Iceland spar with edges parallel to the crystalline axis, and united with their faces in opposite directions, with an interposed transparent medium (Canada balsam) of such refractive index that the Ordinary ray is totally reflected at its surface, while the Extraordinary ray is transmitted. See (115) &c.

parallel, whatever be their position, the fringes of interference are seen well, and the dark bars are perfectly black. If they are not parallel, the dark bars are not so black; and if they are at right angles to each other, the fringes disappear entirely. It appears therefore that pencils of light polarized in planes at right angles to each other cannot interfere (that is cannot destroy each other) in circumstances in which pencils of common light or pencils of light polarized in the same plane, can destroy each other.

99. From the experiments that we have described, the following general conclusions are drawn.

(1) If from common light we produce, by any known contrivance, light that is polarized in one plane, there is always produced at the same time light more or less polarized in the plane perpendicular to the former.

(2) Light polarized in one plane cannot be made to furnish light polarized in the perpendicular plane.

(3) Light polarized in one plane cannot be destroyed by light polarized in the perpendicular plane.

The first leads at once to the presumption that polarization is not a modification or change of common light, but a resolution of it into two parts equally related to planes at right angles to each other; and that the exhibition of a beam of polarized light requires the action of some peculiar forces (either those employed in producing ordinary reflection and refraction or those which produce crystalline double refraction) which will enable the eye to perceive one of these parts without mixture of the other. This presumption is strongly supported by the phænomena of partially-polarized light. If light falls upon a plate of glass inclined to the ray, the transmitted light, as we have seen, is partially polarized. If now a second plate of glass be placed in the path of the transmitted light, inclined at the same angle as the former plate, but with its plane of reflection at right angles to that of the former plate, the light which emerges from it has lost every trace of polarization; whether it be examined only with the analyzing plate B, or by the interposition of a plate of

[illegible] manner to be explained hereafter (145). This [illegible] only on the supposition that the effect of the [illegible] glass was to diminish that part of the light which [illegible] to one plane (without totally removing it), and [illegible] of the second plate is to diminish in the same [illegible] that part of the light which has respect to the other [illegible] therefore that, after emergence from the second [illegible] the two portions of light have the same proportion as [illegible] On considering this presumption in conjunction with [illegible] and third conclusions, we easily arrive at this simple hypothesis explaining the whole:

[illegible] light consists of undulations in which the vibrations of each particle are in the plane perpendicular to the [illegible] of the wave's motion. The polarization of light is [illegible] of the vibrations of each particle into two, one [illegible] a given plane passing through the direction of the [illegible] and the other perpendicular to that plane; which [illegible] that we shall not allude to at present), become in [illegible] the origin of waves that travel in different directions. When we are able to separate one of these from the [illegible] say that the light of each is polarized. When the [illegible] vibration parallel to the plane is preserved unaltered, [illegible] perpendicular to the plane is diminished in a given [illegible] versa), and not separated from it, we say that [illegible] partially polarized.

The reader who has possessed himself fully of this hypothesis, will see at once the connection between all the experiments given above.

1[illegible]. For the general explanation of these experiments, and for the accurate investigation of most of the phænomena [illegible] after described, it is indifferent whether we suppose the vibrations constituting polarized light to take place parallel to the plane of polarization, or perpendicular to it. There are reasons however, connected with the most profound investigations* into the nature of crystalline separation and into the nature of reflection from glass, &c., and confirming

* This question has been discussed carefully, and with some difference of [illegible] by Fresnel, Cauchy, Professor Stokes, and others. In the text, we [illegible] the opinion of Fresnel and Stokes.

each other in a remarkable degree, that incline us to choose the latter: and thus,

When we say that light is polarized in a particular plane, we mean that the vibration of every particle is perpendicular to that plane.

Thus, in the undulation constituting the Ordinary ray of Iceland spar, the vibration of every particle is perpendicular to the principal plane of the crystal: in that constituting the Extraordinary ray, the vibration of every particle is parallel to the principal plane. When light falls upon unsilvered glass at the polarizing angle, the reflected wave is formed entirely by vibrations perpendicular to the plane of incidence: the transmitted wave is formed by some vibrations perpendicular to the plane of incidence, with an excess of vibrations parallel to the plane of incidence.

101. The reader will perceive that it is absolutely necessary to suppose, either that there are no vibrations in the direction of the wave's motion, or that they make no impression on the eye. For if there were such, there ought in the experiment of (98) to be visible fringes of interferences: of such however there is not the smallest trace.

102. As we now suppose light generally to consist of two sets of vibrations which cannot interfere with each other, it becomes important to establish some measure of the intensity of the compound light. It seems that this cannot be any other than the sum of the intensities corresponding to the two sets of vibrations. So that if the displacement from one vibration be represented by $a \sin(vt - x + A)$, and that from the other by $b \sin(vt - x + B)$, the intensity of the mixed light will be $a^2 + b^2$. This then is the expression which we ought in strictness to have used in our former investigations. But as in all these (except those relating to reflection from plane glasses and lenses) the quantities a and b have in every part of the operation the same proportion, it is evident that the results, considered as giving the proportion of intensities of light, are in every instance correct.

PROP. 21. To explain on mechanical principles the trans-

motion of a wave in which the vibrations are transverse to the direction of its motion.

103. In fig. 25 let the faint dots represent the original situations of the particles of a medium, arranged regularly in rows, each line being at the distance h from the next. Suppose all the particles in each vertical line disturbed vertically by the same quantity; the disturbances of different vertical lines being different. Let x be the horizontal abscissa of the second row; $x-h$ that of the first, and $x+h$ that of the third: let u, $u_{,}$ and u' be the corresponding disturbances. The motions will depend upon the extent to which we suppose the forces are sensible. Suppose the only particles whose forces on A are sensible, to be

$$B, \quad C, \quad D, \quad E, \quad F, \quad G,$$

omitting those in the same line, as their attractions are equal and in opposite directions: and suppose them to be attractive and as the inverse square of the distance: and the absolute force of each $= m$. The whole force tending to pull A downwards is

$$\frac{m(h-u-u_{,})}{\{h^2+(h-u-u_{,})^2\}^{\frac{3}{2}}}+\frac{m(u-u_{,})}{\{h^2+(u-u_{,})^2\}^{\frac{3}{2}}}-\frac{m(h-u+u_{,})}{\{h^2+(h-u+u_{,})^2\}^{\frac{3}{2}}}$$

$$-\frac{m(h-u-u')}{\{h^2+(h-u-u')^2\}^{\frac{3}{2}}}+\frac{m(u-u')}{\{h^2+(u-u')^2\}^{\frac{3}{2}}}-\frac{m(h-u+u')}{\{h^2+(h-u+u')^2\}^{\frac{3}{2}}}.$$

Expanding these fractions, and neglecting powers of $u-u_{,}$ and $u-u'$ above the first, the force tending to diminish u is

$$\left(1-\frac{1}{2^{\frac{3}{2}}}\right)\frac{m}{h^3}(2u-u_{,}-u').$$

Putting for $u_{,}$

$$u-\frac{du}{dx}h+\frac{d^2u}{dx^2}\cdot\frac{h^2}{2},$$

and for u'

$$u+\frac{du}{dx}h+\frac{d^2u}{dx^2}\cdot\frac{h^2}{2},$$

we find

$$\frac{d^2u}{dt^2}=\left(1-\frac{1}{2^{\frac{5}{2}}}\right)\frac{m}{h}\cdot\frac{d^2u}{dx^2}\,*$$

an equation of exactly the same form as that for the transmission of sound (10). The solution therefore has the same form: and therefore the transversal motion of particles sup-

* If h is so large with regard to the length of a wave that the terms after h^2 cannot be safely neglected, we may, by assuming a form for the function expressing u, integrate the equation

$$\frac{d^2u}{dt^2}=-\left(1-\frac{1}{2^{\frac{5}{2}}}\right)\frac{m}{h^3}\,(2u-u_{,}-u').$$

If, as we usually suppose,

$$u=A\sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\},$$

$$\text{then }\frac{d^2u}{dt^2}=-\frac{4\pi^2v^2}{\lambda^2}A\sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\},$$

$$u_{,}+u'=A\sin\left\{\frac{2\pi}{\lambda}(vt-x+h)\right\}$$

$$+A\sin\left\{\frac{2\pi}{\lambda}(vt-x-h)\right\},$$

$$=2A\,.\cos\frac{2\pi h}{\lambda}\,.\sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\},$$

$$\text{and } 2u-u_{,}-u'=4A\,.\sin^2\frac{\pi h}{\lambda}\,.\sin\left\{\frac{2\pi}{\lambda}(vt-x)\right\};$$

and by substitution, the equation becomes

$$-\frac{4\pi^2v^2}{\lambda^2}=-4\left(1-\frac{1}{2^{\frac{5}{2}}}\right)\cdot\frac{m}{h^3}\cdot\sin^2\frac{\pi h}{\lambda};$$

$$\text{whence } v^2=\left(1-\frac{1}{2^{\frac{5}{2}}}\right)\cdot\frac{m}{h}\cdot\frac{\lambda^2}{\pi^2h^2}\sin^2\frac{\pi h}{\lambda},$$

$$\text{and } v=\sqrt{\left\{\left(1-\frac{1}{2^{\frac{5}{2}}}\right)\cdot\frac{m}{h}\right\}}\,\frac{\sin\frac{\pi h}{\lambda}}{\frac{\pi h}{\lambda}}.$$

This expression increases as λ increases; that is, undulations consisting of long waves travel with greater velocity than those consisting of short waves. Thus the different refrangibility of differently coloured rays is accounted for. See Article 38. For other modifications of this theory, and their comparison with the observed indices of refraction of different rays in different media, the reader is referred to Professor Powell's treatise *On the Undulatory Theory as applied to the explanation of unequal refrangibility.*

motion also follows the same law as the direct motion of the particles of air: that is, it follows the law of undulation.

144. It seems probable that if we had supposed any other regular arrangement, or taken any other law of force, the same conclusion would have been obtained. And if we suppose the arrangement symmetrical with respect to certain fixed lines, but different in distance of particles, &c. in different directions (as for instance if we suppose every eight adjacent particles to be at the angles of a parallelopiped as in fig. 26), then for vibrations of the particles in different directions the multiplier of $\frac{d^2u}{dx^2}$ will be different, and consequently the velocity of transmission of the wave (which is the square root of the multiplier) will be different. And the velocities of two waves may be different even when they are going in the same direction, provided that one of these waves consists of vibrations in one direction, and the other of vibrations in another direction, as if for instance in fig. 26 the directions of both waves were perpendicular to the paper, but if one set of vibrations were in the direction up and down, and the other in the direction right and left. For the force with which the particles act on each other depends on the distance of the particles in the direction of the waves' motion, and on their distance in the direction of the particles' vibration: and in the case supposed, the latter element is different for the two waves, though the former is the same.

145. If the displacement of a particle, considered as in any direction, be resolved into three displacements in the directions of x, y, z, the variations of force in those directions produced by the alteration of a single particle (and consequently the force produced by the whole system) are the same as if the displacements in those directions had been made independently. From this it easily follows that the sum of any number of displacements causes forces equal to the sum of the forces corresponding to the separate displacements: and then, by the reasoning in (10) and (11), any number of undulations, produced by vibrations in different directions, may co-exist without disturbing each other.

Prop. 22. To explain the separation of common light into

two pencils by doubly-refracting crystals: and to account for the polarization of the two rays in planes at right angles to each other.

106. We shall assume for the state of the particles of ether within a crystal, an arrangement similar to that described in (104), or at least possessing this property, that there are three directions* at right angles to each other, in which if a particle be disturbed, the resultant of the forces acting on it will tend to move it back in the same line in which the displacement is produced. These lines we suppose to be parallel to some lines determined by the form of the crystal.

107. Now in general the displacement† of a particle or a series of particles will not produce a force whose direction coincides with the line of displacement. For suppose the disturbance in the direction of x to be X; that in the direction of y to be Y: and suppose the corresponding forces to be a^2X and b^2Y. The tangent of the angle made by the resultant force with the axis of x is $\frac{b^2Y}{a^2X}$: but the tangent of the angle made by the direction of displacement with the axis of x is $\frac{Y}{X}$: and these are different if a^2 and b^2 are different. In the same manner if we supposed a displacement Z in the direction of z, and if it produced a force c^2Z, the tangents of the angles, made by the projection of the resultant's direction

* M. Fresnel has demonstrated that this must be the case when the small displacement of a particle in the direction of any one of the co-ordinates produces forces in the direction of all, represented by multiples of that displacement. This is apparently the most general supposition that can be made. *Mémoires de l'Institut*, 1824. See also Griffin's *Theory of Double Refraction.*

† We have spoken here of *displacements* as if the forces concerned in the transmission of a wave were thus put in play by *absolute* displacements. It is however plain from (103) that the forces on A really put in play are produced by *relative* displacements: but it is evident that these forces are the same as those that would be put in play by the *absolute* displacement

$$\tfrac{1}{2}(2u - u_{,} - u') \text{ or } \frac{d^2u}{dx^2}\cdot\frac{h^2}{2}.$$

In like manner, when the direction of displacement is any whatever, the quantity $\frac{1}{2}(2u - u' - u_{,})$ in its proper direction may be resolved into the direction of the co-ordinates, and the forces really acting on A will be the forces corresponding to these spaces considered as *absolute* displacements.

on the planes of xz and yz with the axis of z, would be $\dfrac{a^2X}{c^2Z}$ and $\dfrac{b^2Y}{c^2Z}$: while those made by the projection of the line of displacement would be $\dfrac{X}{Z}$ and $\dfrac{Y}{Z}$.

108. Now suppose that, in fig. 26, MN is the front of a wave: or by the definition of (20) and the assumptions of (99) and (101), all the particles in the plane of which MN is the projection are moving with equal motions in that plane. In general the force which acts on these particles in consequence of their displacement, is not in the direction of the displacement, and is not even in the plane MN. Resolve it into two, one perpendicular to the plane and one parallel to it. The former of these may be neglected, because it can only produce a motion which, by (101), is not sensible to the eye. The latter, though in the plane, will not generally be in the direction of the displacement. It is impossible then to find what motions will be caused by this displacement without resolving it. If we can resolve it into two, such that the force produced by each displacement is in the direction of that displacement, then we can find for each of these separately the motions that will result from it. It is clear now that we have fallen on a case exactly similar to that of (104), and the conclusion is the same, namely, that there will be two series of waves travelling with different velocities.

109. Now in (34) we have found that the refraction in a transparent medium depends on the velocity of the wave within that medium. Consequently the refraction of the two series of waves will be different; and thus is explained the bifurcation of the ray, when common light is incident on a surface of the crystal. And each of these consists of vibrations parallel to one line, that is, by (99), of polarized light: and, as will appear by subsequent investigations, the lines of vibration are perpendicular to each other, and therefore the planes of polarization (which are perpendicular to the lines of vibration) are perpendicular to each other. This explanation may be considered as the most important step in Physical

Science since the establishment of the law of gravitation by Newton.

PROP. 23. To investigate the law of double refraction in uniaxal crystals.

110. By *uniaxal crystals* we mean those in which $b^2 = a^2$, while c^2 is not equal to a^2. The investigation, it is seen from (108) and (109), reduces itself to these two things; the discovery of those directions of displacement in the plane of a wave in which the resolved part of the force parallel to the plane is in the same direction as the displacement, and the investigation of the velocity of transmission for waves whose vibrations are in those directions. Now the force, produced by a displacement in any direction parallel to the plane of xy, is in the same direction as the displacement: and therefore it is indifferent what line in the plane of xy we take for x. Let x then be perpendicular to the intersection of the front of the wave with the plane xy. In fig. 27, let MN be the projection on the paper of the front of the wave (supposed perpendicular to the paper), AM the axis of x, AN the axis of z, which we shall call the axis* of the crystal: θ the angle made by the front of the wave with the plane of xy. Then it is plain, from the symmetry of the forces with respect to z, that a displacement parallel to the line MN will cause a force whose resolved part parallel to the plane MN is in the line MN; and that a displacement in the plane MN perpendicular to the line MN will cause a force also perpendicular to MN. The vibration then of the wave incident on the crystal must be resolved into two vibrations parallel to these, and these vibrations, as in (108), will produce two rays that will travel with different velocities.

111. Now the force put in play by a displacement perpendicular to the paper is represented by $a^2 \times$ displacement. Consequently the wave depending on these vibrations moves with the velocity a whatever be the position of the front of the wave. This is the same law as that assumed in (34)

* This always coincides with the mineralogical axis of the crystal. Thus in Iceland spar it is in the solid angle included by three obtuse angles of the planes of cleavage, and makes equal angles with them; in quartz, tourmaline, beryl, &c. it is the axis of the prism.

for common refracting media, and the resulting law of refraction is therefore the same. Let any plane passing through the axis of z or the axis of the crystal be called a *principal plane* of the crystal; then this conclusion may be stated thus: the waves consisting of vibrations perpendicular to a principal plane of the crystal are refracted according to the ordinary law of refraction. This accounts for the refraction and polarization of the ordinary ray.

112. For the displacement in the plane of the paper: putting D for that displacement, it may be resolved into $D \cos\theta$ parallel to a, and $D \sin\theta$ parallel to z. The resulting forces will be represented by $a^2 D \cos\theta$ parallel to x and $c^2 D \sin\theta$ parallel to z; and the sum of the resolved parts of these parallel to MN is represented by

$$D\,(a^2 \cos^2\theta + c^2 \sin^2\theta).$$

The velocity of transmission of the wave perpendicular to its own front is therefore

$$\sqrt{(a^2 \cos^2\theta + c^2 \sin^2\theta)}.$$

This is not the same in all directions; and hence the waves consisting of vibrations parallel to a principal plane of the crystal are not refracted according to the ordinary law.

113. If now the front of a wave produced by such vibrations have at any time the form PQR, fig. 28, the form of the front at a succeeding time will be determined by taking Pp perpendicular to the surface at P and proportional to the value of

$$\sqrt{(a^2 \cos^2\theta + c^2 \sin^2\theta)} \text{ there;}$$

Qq perpendicular to the surface at Q and having the same proportion to the value of

$$\sqrt{(a^2 \cos^2\theta + c^2 \sin^2\theta)} \text{ there; \&c.}$$

If the wave were produced originally by an agitation at C, all the successive fronts must be similar; and if we take points of all where their tangents are parallel, that is, points along a radius CQ, the perpendicular distance of each front from the next is proportional to

$$\sqrt{(a^2 \cos^2\theta + c^2 \sin^2\theta)},$$

and therefore the sum of all, which is the same as the perpendicular on the tangent at Q, must be proportional to

$$\sqrt{(a^2 \cos^2 \theta + c^2 \sin^2 \theta)}.$$

Therefore, to find the form of an extraordinary wave diverging from a point, we must solve this problem: To find the curve where the perpendicular on the tangent is proportional to

$$\sqrt{(a^2 \cos^2 \theta + c^2 \sin^2 \theta)},$$

θ being the angle made by the tangent with the axis of x. It is well known that this is an ellipse, whose axes in the directions of z and x are in the proportion of $a : c$. Consequently, to discover the path of the extraordinary ray, we must suppose the waves produced by vibrations parallel to a principal plane to diverge in the form of a spheroid of revolution round a line parallel to the axis of z, and must suppose the semi-axes of the spheroid parallel and perpendicular to z to be represented by a and c: and must then proceed as for common light. The radius of the sphere into which the ordinary wave has diverged must at the same time be represented by a.

114. It is easily seen that the motion of an extraordinary wave in the crystal is not generally perpendicular to its front. For let AB, fig. 29, be an aperture through which a small part of an extraordinary wave passes: CD a line parallel to the axis of the crystal. Consider A, a, b, c, &c. as the origins of equal spheroidal waves, the axes of the waves being parallel to CD. It is plain that the part between E and F is the only place in which the waves strengthen each other, as at all points on both sides of this they precede or follow each other by different quantities, and therefore mutually destroy each other, while between E and F the neighbouring waves meet in nearly the same phase. The wave therefore will seem to travel from AB to EF. The general rule therefore is this; describe a spheroid whose axis is parallel to the axis of the crystal, and find the point of its surface where the tangent plane is parallel to the front of the wave; then the motion of the wave is parallel to the radius of that point.

115. The general construction for determining the path

of both rays is this. In fig. 30, let the plane of the paper be the plane of incidence, BA' the projection of the surface of the crystal, AB the front of a wave moving in the direction AA'. Let CD be the axis of the crystal, not necessarily in the plane of the paper. While a part of the wave moves in vacuum from A to A', suppose that the ordinary wave diverging from B will spread into the sphere Fo, and the extraordinary wave into the spheroid Fe (whose axis of revolution = diameter of sphere). Through the line, of which A' is the projection, draw a plane touching the sphere in o; this plane is the front of the ordinary wave, and Bo represents the direction and velocity of its motion. Through the same line draw a plane touching the spheroid in e; this plane is the front of the extraordinary wave, and Be represents the direction and velocity of its motion. If the axis of the spheroid does not lie in the plane of the paper, and is not perpendicular to the plane of the paper, the point e will not be in the plane of the paper: and thus the direction of the extraordinary ray will not lie in the plane of incidence. The demonstration of this construction is exactly similar to that of (34).

The course of an extraordinary ray after internal reflection is to be found in a manner analogous to that of (32). Thus in fig. 30, suppose that the extraordinary wave whose front is $A'e$ moves in the direction $A'G$ and is wholly or partially reflected at the surface GH. When the part A' has arrived at G, suppose the part e to be at I on its way to H. Then when I reaches H, the small wave caused by the disturbance at G will have extended into a spheroid similar and equal to that which must be described from the center I to pass through H. Let KH be this spheroid (the axis being always parallel to CD), and LM the spheroid equal to it whose center is G: let HL be the tangent plane passing through the line projected in H. Then, as in (32), HL is the front of the reflected wave, and, as above, GL is the direction of the reflected ray. Here the angle of reflection is not generally equal to the angle of incidence, and the angles of incidence and reflection are not generally in the same plane.

116. If we consider the extent of refraction to be deter-

mined by the change in the position of the front of the wave (which is sometimes the most convenient way), and if the spheroid be oblate, as it is for Iceland spar, beryl, tourmaline, &c., the extraordinary ray is always less refracted than the ordinary ray, since in fig. 30 the spheroid includes the sphere. If the spheroid be prolate, as in quartz, uniaxal apophyllite, &c. the extraordinary ray is always more refracted than the ordinary ray. The normal to the front of the wave is always in the plane of incidence.

117. A compound prism which produces great angular separation of the two rays is thus constructed. Let a prism A be cut from Iceland spar with its edge parallel to the axis, and another prism B of equal angle with its edge perpendicular to the axis, and let them be placed as in fig. 31. The vibrations parallel to the plane of the paper will furnish the ordinary ray of A and the extraordinary ray of B; that is, this wave will be most refracted by A towards C, and least by B towards D, and it will therefore on the whole pass towards C. In a similar manner, the wave produced by vibrations perpendicular to the paper will be least refracted by A towards C, and most refracted by B towards D, and it will therefore on the whole pass towards D. If the prisms be cut from quartz, the separation is in the opposite direction; it is smaller also, as the prolate spheroid of quartz differs less from a sphere than the oblate spheroid of Iceland spar.

PROP. 24. To investigate the law of double refraction in biaxal crystals.

118. By *biaxal crystals* are meant those in which a^2, b^2, c^2, are all different. Our limits will not allow us to go through the whole of this investigation, and we shall merely give the principal steps, referring for details to the *Mémoires de l'Institut*, 1824; the *Annales de Chimie*, 1828; the *Cambridge Transactions*, Vol. VI. p. 85; and Mr Griffin's *Theory of Double Refraction;* and generally to the *Memoirs* of the principal scientific societies, especially those of the Royal Irish Academy, to the *Cambridge Mathematical Journal*, and to the *Philosophical Magazine*.

119. The first thing to be done, as in (108), is to find two directions in the front of a plane wave in which a displacement produces a force in the same direction, neglecting that force which is perpendicular to the front. As we shall only have to calculate the forces in the directions possessing this property, we shall at once resolve the whole force of displacement into two, one parallel to the direction of displacement, the other perpendicular to it (not necessarily perpendicular to the front of the wave), and shall neglect the latter. If the direction of displacement make angles X, Y, Z, with the axes of x, y, z, this resolved force, as in (111), is

$$\text{displacement} \times (a^2 \cos^2 X + b^2 \cos^2 Y + c^2 \cos^2 Z).$$

Construct a surface of which the latter factor is the radius, which we shall call the surface of elasticity; it is easily seen that the radius is the squared reciprocal of the radius in the ellipsoid whose axes are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

120. Make a section by the plane front of the wave through the center of this surface; the radius vector of the section will be the square of the reciprocal of the radius vector in the corresponding section of the ellipsoid, that is in an ellipse; and this section of the surface of elasticity will therefore be a curve symmetrical with respect to its greatest and least diameters, which are at right angles.

121. The radius vector of this section in any direction represents the resolved part, in that direction, of the force produced by displacement in that direction, the neglected part being perpendicular to that direction and not necessarily perpendicular to the front of the wave. If now we examine the direction of displacement in which the neglected part is perpendicular to the front of the wave, it is found that the greatest and least diameters above alluded to are the only ones which satisfy this condition. Consequently the vibrations must be resolved into two, parallel respectively to these diameters; and these will produce the two waves. Their velocities will be represented by the square roots of the values of those semi-diameters.

122. In two positions of the front of the wave, and no more, the section becomes a circle. Whatever then is the direction of vibration in that front, the velocity of transmission is the same, and there is no separation of fronts of waves, though there may be separation into two or many rays. The two lines perpendicular to these circles are called the *optic axes*.

123. The difference between the squares of the velocities of the two waves is proportional to the product of the sines of the two angles made by the front of the wave with the two circular sections, or to the product of the sines of the angles made by the normal to the front with the two optic axes.

124. The plane of polarization of one ray bisects the angle made by the two planes which pass through the normal and the two optic axes: and the plane of polarization of the other is perpendicular to it. This is easily shewn thus: where the front of the wave cuts the two circular sections, the radii in the section by the front must be equal to the radii of the circles, and therefore must be equal to each other, and therefore must make equal angles on both sides of the longest or shortest diameter: and therefore if the planes be projected on a sphere concentric with the surface of elasticity, the point which is the projection of the longest or shortest diameter will bisect one side of the spherical triangle. Construct the spherical triangle whose angles are the poles of those sides: then the circle drawn from the bisection to the pole of that side (which is the projection of one plane of vibration) will bisect the angle made by the sides joining that pole, or the pole of the front, with the optic axes, inasmuch as those sides are equally inclined to the quadrants joining that pole with the two angles of the first triangle where the front of the wave meets the circular sections.

125. The form into which the wave must be supposed to diverge is determined as in (113), by finding the forms of the surfaces where the perpendicular to the tangent plane is proportional to one of the velocities found in (121). After a very troublesome algebraical process it is found that the equation

expressing the two surfaces (which are in fact one continuous surface) is

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2 \\ -b^2(a^2+c^2)y^2-c^2(a^2+b^2)z^2+a^2b^2c^2=0.$$

This cannot be resolved into factors, and therefore cannot express a sphere and any other surface, as in (111) and (113). Consequently neither of the rays is subject to the law of ordinary refraction. This conclusion might also have been drawn from the observation that neither of the velocities found in (121) is constant. The direction, &c. of the two rays when light is incident on a surface of the crystal are found exactly as in (114), using the surface above mentioned instead of the sphere and spheroid, and finding the two positions of the tangent plane passing through the line projected in A', fig. 30.

The properties of this wave-surface are however so important, and one of the deductions from them so singular, that we shall devote some subordinate articles to a statement of them, referring for complete discussion to the *Transactions of the Royal Irish Academy*, Vol. XVII.

125 (*a*). If in the equation above we make $z=0$, we find for the section in the plane of xy

$$(x^2+y^2)(a^2x^2+b^2y^2)-c^2(a^2x^2+b^2y^2)-a^2b^2(x^2+y^2)+a^2b^2c^2=0;$$

or $$(x^2+y^2-c^2)(a^2x^2+b^2y^2-a^2b^2)=0.$$

That is, the curve of intersection of the plane of xy with the surface is in fact two separate curves: one a circle whose radius is c, the other an ellipse whose semi-axes are a and b. In like manner, the curves of intersection with yz will be, a circle whose radius is a, and an ellipse whose semi-axes are b and c: and the curves of intersection with xz will be, a circle whose radius is b, and an ellipse whose semi-axes are a and c.

125 (*b*). Suppose now that a is greater than b, and b greater than c. Then, on the plane yz, the circle of radius a will completely inclose without touching the ellipse of semi-axes b and c: and, on the plane xy, the ellipse of semi-axes

a and b will completely inclose without touching the circle of radius c. But on the plane xz, the ellipse of semi-axes a and c will cut the circle of radius b at four points. These propositions are due to Fresnel.

125 (c). When the nature of the wave-surface is examined with particular reference to these four points (which was first done by Sir W. R. Hamilton) it is found that, as viewed from the outside, there are four conical depressions; and, as viewed from the inside or center, there are four cones projecting towards the outside. The surface, in fact, consists of two quasi-ellipsoidal sheets or surfaces, one completely inclosed within the other; but at these four points, the inner surface is drawn outwards, and the outer surface drawn inwards, so as to establish a connexion between the two surfaces.

125 (d). On examining the nature of the ring (very nearly circular) which forms the external base of any one of the depressed cones, it is found that it is in one plane: that, in fact, at that part, a tangent-plane touches the external surface in a ring; and that the plane of the ring and tangent is parallel to one of the circular sections in (122).

125 (e). Remarking then the construction for determining the course of a refracted ray in (114) and (115), it will be seen that, if a wave of light enter the crystal with its front normal to the optic axis, or parallel to the circular sections of the surface of elasticity (122), or parallel to the tangent-plane of the conical depression; its front will continue parallel to that tangent-plane; and the direction of the ray will be the direction of a line from the center of the wave-surface to some point of the tangent-ring. At first, there appears to be no reason why the course of the ray should select one point of the ring rather than another; this is determined by the following consideration.

125 (f). A construction similar to that of (124) determines the plane of polarization of the ray. It seems that*,

* For a ray which falls *exactly* in any part of the ring, the wave-front is *exactly* normal to the optic axis. But for a ray which falls *by the smallest quantity* external (for instance) to the ring, the front of the wave is inclined as

in this case, (in which the construction of (124) fails), the plane of polarization is thus determined: Draw one plane through the ray and through one optic axis, and draw another plane through the ray and through the other optic axis; and the plane of polarization bisects the angle formed by these two planes. To one plane of polarization thus determined, there corresponds only one ray. Consequently, if polarized light be incident, only one ray is formed: but if common light be incident (which, not improbably, consists of successive series of waves polarized in every conceivable plane), rays will be formed directed to every point of the ring, each ray having the polarization proper to its point of the ring; and a conical sheet of light will be formed within the crystal. On emerging at a plane surface parallel to the surface of entrance, the emergent rays will be parallel to the first incident ray; and a cylindrical sheet of light will be formed in air, still preserving in every part its peculiar polarization.

125 (*g*). This very singular result of the theory of undulations has been verified by Dr H. Lloyd, as regards both the distribution of light and its polarization. And the agreement of the prediction and the observation is undoubtedly one of the most remarkable proofs of the general correctness of the considerations by which the laws of double refraction are determined.

125 (*h*). If a ray of light, consisting of light polarized in different planes, be made to pass through the crystal in the direction of the line from the center of the wave-surface to the vertex of the cones (which will be done by permitting light from all directions to fall upon the crystal, and limiting its course through the crystal by plates with very small holes on its opposite sides), each differently polarized beam will re-

from the center of the ring. Applying this consideration to the construction of (124); and remarking that the intersection of the front of the wave with the nearest circular section will be determined by bisecting the space between those two intersections; it is easily seen that the rotation of the plane of polarization is half as rapid as the rotation of the intersection with the near circular section; that is, half as rapid as the revolution of the ray round the ring. This amounts nearly to the construction in the text.

ceive a different refraction at the second surface, and an external conical sheet of light will be formed. This also is verified by experiment.

*125. Before leaving this investigation we must remark that this theory is imperfect in the same degree as the explanation of refraction. In every uniaxal crystal, we believe, the axis is the same for all the colours, but the ratio of a to c is not the same for different colours. In biaxal crystals generally the direction of the three axes is the same for different colours, but the ratio of a, b, c, is not the same, and consequently the position of the optic axes (122) is not the same for different colours, though the optic axes for all colours are in the same plane. And it has been discovered by Sir John Herschel that the direction of the three axes is in some instances different for different colours, and then the optic axes for different colours are not all in the same plane.

PROP. 25. Light polarized in the plane of incidence falls on a refracting surface of glass, &c.: to find the intensity of the reflected and the refracted ray.

126. The three next investigations which we offer to the reader cannot be considered as wholly satisfactory. The extreme difficulty of mathematical investigation into the state of particles at the confines of two media prevents us from making them more complete. It is however gratifying to know that they are fully supported by experiment, and that they have given a law to phænomena, of which some appeared inexplicable, and others would never have been reduced to laws by observation alone.

127. Suppose that the particles of ether, retaining the same attractive force†, are in the inside of glass, &c. loaded with some matter which increases their inertia in the ratio of $1 : n$, without increasing their attraction. The equation of (103) would be changed to this:

$$\frac{d^2u}{dt^2} = \frac{1}{n}\left(1 - \frac{1}{2^{\frac{1}{2}}}\right) \cdot \frac{m}{h} \cdot \frac{d^2u}{dx^2} \cdot$$

† Perhaps this supposition is hardly reconcileable with that made in the last proposition.

If the solution before were $u = \phi(vt - x)$, the solution would now be

$$u = \phi(vt - x\sqrt{n}).$$

The velocity of transmission is diminished therefore in the ratio of $\sqrt{(n)} : 1$. But we have supposed that the velocity is diminished in the ratio of $\mu : 1$. Consequently $n = \mu^2$.

128. Now suppose that we have a series of equal quantities of the ether in a line, and that a transverse motion is given to the first, which, from the constitution described in (103) it has the power of transmitting to the second, &c. When we arrive at the surface of the glass, we must take volumes of the denser ether, whose dimensions are determined in the direction of the transmission of the wave by lengths proportional to the velocity of transmission, and in the other directions by their correspondence with the quantity of ether which puts them in motion. Thus in fig. 32, if $DF = \frac{BD}{\mu}$, the ether in $ABDC$ may be considered as putting $CDFE$ in motion. Put i for the angle of incidence, i' for that of refraction. The proportion of the lengths in the direction of the ray is $\mu : 1$, or $\sin i : \sin i'$. The proportion of the breadths is $\cos i : \cos i'$. The proportion of densities is $1 : \mu^2$, or $\sin^2 i' : \sin^2 i$. Combining these proportions, the proportion of the masses is

$$\sin i' . \cos i : \sin i . \cos i'.$$

Now if an elastic body impinges on an equal elastic body, it loses its own velocity and communicates to the other a velocity equal to its own: this is similar to the action of one mass of the ether in vacuum on the next. Supposing the similarity of action to apply to the different states of ether at the confines at the medium, we must compare this with the motion of two unequal elastic bodies A and B after the impact of A with the velocity V on B originally at rest. It is known that A retains the velocity $\frac{A - B}{A + B} V$, and that B receives the velocity $\frac{2A}{A + B} V$. Substituting for A, $\sin i' . \cos i$, and for B,

$\sin i . \cos i'$, we find for the motion retained by the external ether, $\frac{\sin (i'-i)}{\sin (i'+i)} \times$ its previous motion; and for that communicated to the internal ether, $\frac{2 \sin i' . \cos i}{\sin (i'+i)} \times$ previous motion of external ether. Now by a succession of numerous impulses of this kind, following a given law, a series of waves with any law of displacement may be produced: and every impulse produces parts in the two media having the proportions given above. If then the original displacement be represented by

$$a \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

that retained by the external ether, and which produces the reflected ray, must be

$$a \frac{\sin (i'-i)}{\sin (i'+i)} \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

and that transmitted to the internal ether, and which produces the refracted ray, must be

$$a \frac{2 \sin i' . \cos i}{\sin (i'+i)} \sin \left\{\frac{2\pi}{\lambda}(vt - \mu x)\right\}.$$

These formulæ apply equally to refraction from air into glass, and from glass into air, giving i and i' their proper values. The intensities of the rays will be represented by the squares of the coefficients.

PROP. 26. Light polarized perpendicular to the plane of incidence falls on a refracting surface: to find the intensity of the reflected and the refracted ray.

129. We cannot here use the same kind of reasoning as in (128), because the motion of displacement (being in the plane of incidence and perpendicular to the path of the ray) is not in the same direction for any two of the three rays. To overcome this difficulty, M. Fresnel has adopted the following hypotheses. First he supposes that the law of *vis viva* holds: that is, that the sum of the products of each mass by the square of its velocity is constant. (This is certainly true if as in

the masses are supposed to act nearly as elastic bodies. And in all cases of mechanical action it is equal to the sum of all the integrals of *force* × *space through which it has acted*, which is constant in all the cases of undulation that we can strictly examine, and is probably constant in this.) Next he supposes that the resolved parts of the motion perpendicular to the refracting surface will preserve after leaving the surface the same relation which they have there, and which, if they follow the same laws as those of the impact of elastic bodies, would be thus connected: the relative motions before and after impact will be equal in magnitude but opposite in sign. (This is confessed by M. Fresnel to be purely empirical.) Adopting these hypotheses, and considering the masses to be as

$$\sin i' . \cos i \; : \; \sin i . \cos i';$$

and representing the displacements in the incident, refracted, and reflected ray, (estimated positive in that direction perpendicular to their respective rays which is nearest to that of a body falling perpendicularly from vacuum on the refracting surface,) by a, b, c; we have the following equations:

$$\sin i' . \cos i . a^2 = \sin i . \cos i' . b^2 + \sin i' . \cos i . c^2$$

$$a \cos i = b \cos i' + c \cos i.$$

Eliminating b,

$$(\sin 2i' + \sin 2i)\, c^2 - 2 \sin 2i . ac - (\sin 2i' - \sin 2i)\, a^2 = 0,$$

$$\text{or} \quad (c - a)\left\{(\sin 2i' + \sin 2i)\, c + (\sin 2i' - \sin 2i)\, a\right\} = 0.$$

This equation is satisfied by $c = a$: but that would give $b = 0$, and therefore expresses only total reflection, which would require exactly the same mathematical conditions as those that we have used, but would not correspond to the physical circumstances of the problem now before us. The other is the only solution which we want: it gives

$$c = -a \frac{\tan (i' - i)}{\tan (i' + i)},$$

$$\text{and } b = a \frac{\cos i}{\cos i'} \left\{1 + \frac{\tan (i' - i)}{\tan (i' + i)}\right\}.$$

Hence if the displacement produced by the incident wave is

$$a \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

that produced by the reflected wave is

$$-a \frac{\tan(i' - i)}{\tan(i' + i)} \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

and that by the refracted wave is

$$a \frac{\cos i}{\cos i'} \left\{1 + \frac{\tan(i' - i)}{\tan(i' + i)}\right\} \sin \left\{\frac{2\pi}{\lambda}(vt - \mu x)\right\}.$$

130. One of the most remarkable inferences from this expression is obtained by making $i' + i = 90^\circ$. The displacement produced by the reflected wave is then $= 0$. Suppose now light consisting of transversal vibrations in all directions to be incident at this angle on a surface of glass. Resolve the vibrations into two sets, one parallel to the plane of incidence and the other perpendicular to it. The former (as we have just seen) will furnish no reflected ray: the latter, by (128), will produce a reflected ray. Consequently the reflected light will consist solely of vibrations perpendicular to the plane of reflection. The condition $i' + i = 90$ gives

$$\sin i' = \cos i, \text{ or } \frac{\sin i}{\mu} = \cos i, \text{ whence } \tan i = \mu:$$

which (95) defines the polarizing angle. Thus the angle of incidence at which, according to theory, the vibrations of the reflected ray are entirely perpendicular to the plane of incidence, is the same as the angle at which, in experiment, the reflected ray is entirely polarized in the plane of incidence. And we have found from theory in (111) that the ray of a uniaxal crystal which undergoes the ordinary refraction, and which (94) is said to be polarized in the principal plane, is produced by vibrations perpendicular to the principal plane. These are two reasons which induce us to say, as in (100), that light polarized in a particular plane consists of vibrations perpendicular to that plane.

131. Another remarkable inference is this. If the two

surfaces of a glass plate are parallel, i and i' at the second surface are the same as i' and i at the first. Consequently, if the light reflected from the first surface is polarized, or if $i+i'$ at the first surface $=90^0$, $i+i'$ at the second surface also $=90^0$, and therefore the light reflected internally from the second surface is also polarized. This is true in experiment.

Many investigations applying to these problems are to be found in the *Cambridge Transactions* and other *Transactions*, the *Philosophical Magazine*, and the *Comptes Rendus* of the French Academy.

PROP. 27. Light polarized in a plane inclined by the angle α to the plane of incidence falls on the surface of a refracting medium: to find the position of the plane of polarization of the reflected light.

132. The displacement of a particle of ether before incidence may be represented by

$$a \sin \left\{\frac{2\pi}{\lambda}(vt-x)\right\}$$

in the direction making with the plane of incidence an angle $(90^0-\alpha)$:

and this may be resolved into

$$a \cos \alpha \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt-x)\right\}$$

perpendicular to the plane of incidence,

$$\text{and } a \sin \alpha \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt-x)\right\}$$

parallel to the plane of incidence. And these expressions will apply to the reflected ray, giving x the same alteration in both, and altering the coefficients in the ratios determined in (128) and (129). Hence we shall have after reflection,

Displacement perpendicular to the plane of incidence

$$a \cos \alpha \frac{\sin (i'-i)}{\sin (i'+i)} \sin \left\{\frac{2\pi}{\lambda}(vt-x)\right\}.$$

Displacement parallel to the plane of incidence

$$-a \sin \alpha \frac{\tan (i' - i)}{\tan (i' + i)} \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\}.$$

Since these are in the same ratio whatever be the value of x, it follows that the displacement compounded of these is entirely in one plane, and therefore the reflected light is polarized. And if β is the angle at which the new plane of polarization is inclined to the plane of incidence, or $90^\circ - \beta$ the angle at which the new direction of vibration is inclined to the plane of incidence, we have

$$\cot \beta = \frac{a \cos \alpha \dfrac{\sin (i' - i)}{\sin (i' + i)}}{-a \sin \alpha \dfrac{\tan (i' - i)}{\tan (i' + i)}} = -\cot \alpha \frac{\cos (i' - i)}{\cos (i' + i)},$$

$$\text{or} \quad \tan \beta = -\tan \alpha \frac{\cos (i' + i)}{\cos (i' - i)}.$$

When i and i' are both small, β and α have different signs: this shews that the planes of polarization before and after reflection are inclined* on opposite sides of the plane of incidence. If $i + i' = 90^\circ$, that is, if the angle of incidence is the polarizing angle, $\beta = 0$, or the plane of polarization of the reflected ray coincides with the plane of incidence: and if i be further increased, β and α have the same signs. These results have been verified by numerous observations and careful measures of M. Arago and Sir David Brewster.

Prop. 28. Light is incident on the internal surface of glass at an angle equal to or greater than that of total reflection; to find the intensity and nature of the reflected ray.

133. The expressions in (128) and (129) become impossible. Yet there is a reflected ray, whatever be the nature of the vibrations in the incident light. And on the principle of *vis viva* the intensity of the reflected ray ought to be equal to that of the incident ray, since there is no refracted ray to

* The inclinations are considered to be on the same side when (supposing for facility of conception the angle of incidence to be considerable) the upper parts of both planes are on the same side of the plane of incidence.

consume a part of the *vis viva*. And indeed in the last state of the expressions of (128) and (129) before becoming impossible, that is when $i' = 90^\circ$, each of them becomes $= 1$. After this the expression for the coefficient of vibrations perpendicular to the plane of incidence {putting $\mu \sin i$ for $\sin i'$, and $\sqrt{(-1)} \cdot \sqrt{(\mu^2 \sin^2 i - 1)}$ for $\cos i'$} becomes

$$\frac{\mu \sin i \cdot \cos i - \sin i \sqrt{(-1)} \sqrt{(\mu^2 \sin^2 i - 1)}}{\mu \sin i \cdot \cos i + \sin i \sqrt{(-1)} \sqrt{(\mu^2 \sin^2 i - 1)}},$$

$$\text{or } \cos 2\theta - \sqrt{(-1)} \sin 2\theta,$$

$$\text{where } \tan \theta = \frac{\sqrt{(\mu^2 \sin^2 i - 1)}}{\mu \cos i},$$

and that for the coefficient of vibrations parallel to the plane of incidence becomes

$$\frac{\sin i \cdot \cos i - \mu \sin i \sqrt{(-1)} \sqrt{(\mu^2 \sin^2 i - 1)}}{\sin i \cdot \cos i + \mu \sin i \sqrt{(-1)} \sqrt{(\mu^2 \sin^2 i - 1)}},$$

$$\text{or } \cos 2\phi - \sqrt{(-1)} \sin 2\phi,$$

$$\text{where } \tan \phi = \frac{\mu \sqrt{(\mu^2 \sin^2 i - 1)}}{\cos i}.$$

It is improbable that these formulæ are entirely without meaning: what can their meaning be?

134. M. Fresnel seems to have considered that as the direction of the reflected ray and the nature and intensity of the vibration were already established, there remained but one element which could be affected, namely, the phase of vibration. And it seems not improbable that this may be affected, inasmuch as the incident vibration, though it cannot cause a refracted ray, must necessarily cause an agitation among the particles of the ether outside the glass. It would seem to us most likely that the ray would be retarded (though the phenomena to be hereafter described compel us to admit that it is accelerated): and in all probability differently according to the direction in which the vibrations take place. Nothing then seems more likely than that 2θ and 2ϕ should express these

accelerations*: and as they are angles, they must be combined with the angles in the expression for the vibration. Thus for instance, if

$$a \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

were the expression for the vibrations perpendicular to the plane of incidence on the supposition that they were not accelerated,

$$a \sin \left\{\frac{2\pi}{\lambda}(vt - x) + 2\theta\right\}$$

would be the expression on the supposition that they were accelerated.

135. The only thing which concerns us experimentally is the difference $2\phi - 2\theta$ (which we shall call δ) of the accelerations, for vibrations perpendicular to and parallel to the plane of incidence. Now

$$\tan(\phi - \theta) = \frac{\cos i \sqrt{(\mu^2 \sin^2 i - 1)}}{\mu \sin^2 i},$$

$$\text{whence } \cos\delta = \frac{1 - \tan^2(\phi - \theta)}{1 + \tan^2(\phi - \theta)} = \frac{2\mu^2 \sin^4 i - (1 + \mu^2)\sin^2 i + 1}{(1 + \mu^2)\sin^2 i - 1}.$$

* M. Fresnel's reasoning is of this kind. In several geometrical cases, the occurrence of an imaginary quantity indicates a change of 90^0 in the position of the line whose length is multiplied by $\sqrt{-1}$. It is probable then that here the multiplication by $\sqrt{(-1)}$ denotes that the phase of the vibration which it affects is to be altered (suppose increased) by 90^0. Thus the expression

$$\{\cos 2\theta + \sqrt{(-1)} \sin 2\theta\} . \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

is to be interpreted as signifying

$$\cos 2\theta . \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\} + \sin 2\theta . \sin \left\{\frac{2\pi}{\lambda}(vt - x) + 90^0\right\},$$

$$\text{or } \cos 2\theta . \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\} + \sin 2\theta . \cos \left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

$$\text{or } \sin \left\{\frac{2\pi}{\lambda}(vt - x) + 2\theta\right\}.$$

And similarly for the other.

It appears from this expression that $\delta = 0$ when $\sin i = \frac{1}{\mu}$, or when $\sin i = 1$: and that δ is greatest when

$$\sin^2 i = \frac{2}{1+\mu^2}:$$

the value of $\cos \delta$ being then

$$\frac{8\mu^2}{(1+\mu^2)^2} - 1.$$

If we assume $\delta = 45^0$, we have this equation:

$$\frac{2\mu^2}{(1+\mu^2)\operatorname{cosec}^2 i - \operatorname{cosec}^4 i} = 1 + \sqrt{\tfrac{1}{2}}:$$

the solution of which, supposing $\mu = 1{,}51$, gives

$$i = 48^0.\,37'.\,30'', \text{ or } 54^0.\,37'.\,20''.$$

If then light be incident internally on the surface of crown glass at either of these angles, the phase of the vibrations in the plane of incidence is accelerated more than that of the vibrations perpendicular to the plane of incidence by 45^0. If the light be twice reflected in the same circumstances and with the same plane of reflection, the phase of vibrations in the plane of incidence is more accelerated than that of the other vibrations by 90^0.

136. If then we construct a rhomb of glass, fig. 33, two of whose sides are parallel to the plane of the paper, and the others perpendicular to the paper and projected in the lines *AB*, *BC*, *CD*, *DA*; and if the angles at *A* and *C* are $54^0.37'$: then light incident perpendicular to the end at *F* will be internally reflected at *G* and *H*, making at those points angles of incidence $54^0.37'$, and will emerge at *I* in the direction parallel to that in which it entered at *F*. The immersion at *F* and the emersion at *I* will produce no alteration in the light, but the effect of the two reflections at *G* and *H* will be to accelerate the phases of vibration in the plane of the paper more than those perpendicular to that plane by 90^0. A rhomb thus constructed we shall call *Fresnel's rhomb*.

PROP. 29. Polarized light is internally reflected in a refracting medium at an angle of incidence greater than that necessary for total reflection: to find the nature of the reflected ray.

137. Let the plane of polarization make with the plane of incidence the angle α. Then the vibration, represented by

$$a \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

is performed in a direction making the angle $90° - \alpha$ with the plane of incidence. Consequently the resolved vibrations are

$$a \cos \alpha \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

perpendicular to the plane of incidence, and

$$a \sin \alpha \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

parallel to the plane of incidence. The latter of these, by (135), is more accelerated than the former by δ. If then after reflection we use

$$a \cos \alpha \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

to express the vibration perpendicular to the plane of incidence, we must take

$$a \sin \alpha \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt - x) + \delta\right\}$$

for the vibration parallel to the plane of incidence. The same expression holds if the light is internally reflected any number of times, (the planes of incidence being always the same,) if we take care to give the proper value to δ.

138. Now let us examine the motions of a particle of ether in the reflected pencil. We will take y for the ordinate in the plane of reflection, and z for that perpendicular to it, both measured from the place of rest of the particle, in the plane transverse to the direction of the reflected ray.

(1) Let $\alpha = 45^\circ$, and $\delta = 90^\circ$. (This represents the case of Fresnel's rhomb when the plane of polarization is inclined 45° to that of reflection). Here

$$y = a\sqrt{\tfrac{1}{2}} \cdot \cos\left\{\frac{2\pi}{\lambda}(vt - x)\right\}, \quad z = a\sqrt{\tfrac{1}{2}} \cdot \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

$$\text{and } y^2 + z^2 = \frac{a^2}{2}.$$

That is, every particle describes a circle whose radius is $\dfrac{a}{\sqrt{2}}$.

(2) Let α have any value, δ being $= 90^\circ$. (This is the general case of Fresnel's rhomb). Here

$$y = a \sin\alpha \cdot \cos\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

$$z = a \cos\alpha \cdot \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

$$\text{and } \frac{y^2}{a^2\sin^2\alpha} + \frac{z^2}{a^2\cos^2\alpha} = 1.$$

That is, every particle describes an ellipse, whose semi-axes are $a\sin\alpha$ parallel to the plane of reflection, and $a\cos\alpha$ perpendicular to that plane.

(3) In the general case, α and δ having any values,

$$y = a\sin\alpha\left[\sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\}\cos\delta + \cos\left\{\frac{2\pi}{\lambda}(vt - x)\right\}\sin\delta\right],$$

and $$z = a\cos\alpha \cdot \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\}.$$

Hence $$\sin\frac{2\pi}{\lambda}(vt - x) = \frac{z}{a\cos\alpha}:$$

and $$(y - \tan\alpha \,.\, \cos\delta \,.\, z)^2$$

$$= a^2 \sin^2\alpha \,.\, \sin^2\delta \,.\, \cos^2 \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

$$= a^2 \,.\, \sin^2\alpha \,.\, \sin^2\delta \left[1 - \sin^2 \left\{\frac{2\pi}{\lambda}(vt - x)\right\}\right]$$

$$= a^2 \sin^2\alpha \,.\, \sin^2\delta - \tan^2\alpha \,.\, \sin^2\delta \,.\, z^2:$$

the equation to an ellipse whose axes are inclined to the plane of reflection.

(4) If we compare the expressions for y and z in the first case with the equations to a circular helix (t being considered constant), we find that they exactly coincide. That is, a series of particles which were originally in a straight line, will be at any subsequent time in the form of a circular helix. In the other cases, the position of the particles will be what may by analogy be called an elliptic helix.

(5) For all values of δ, if $\alpha = 0$, or if $\alpha = 90^0$, the reflected light has the same polarization as the incident light.

139. The nature of the light in the reflected ray may then be generally expressed by saying that it is *elliptically polarized:* and in the first case by saying that it is *circularly polarized.* Wherever after this we speak of common polarized light we shall for the sake of distinction call it *plane-polarized* light. From the investigation of the second case it appears that Fresnel's rhomb, by proper adjustment of position with respect to the plane of polarization, is capable of producing elliptically polarized light of every degree of ellipticity. We will therefore suppose that the circularly or elliptically polarized light is produced by Fresnel's rhomb*. For use, it is convenient to have it mounted in a frame which, without stopping the light, admits of its turning round the axis *HI*, fig. 33: this frame may be placed on the board in

* We shall hereafter mention another contrivance which produces nearly but not exactly the same effect, and which has been used more extensively than Fresnel's rhomb.

fig. 24: then the light plane-polarized by A is by the rhomb converted into circularly or elliptically polarized light, and emerges from the end DC opposite to the analyzing plate B in fig. 24. If the mounting be graduated so as to determine the angle made by the plane of polarization with the plane of reflection, then when this angle is 0, 90°, 180°, 270°, the plane-polarized light is not altered: when it is 45°, 135°, 225°, 315°, the emergent light is circularly polarized: when it has any other value, the light is elliptically polarized.

140. Now it is evident that circularly polarized light may be resolved into two vibrations parallel and perpendicular to any arbitrary plane, and that the magnitudes of these vibrations are always the same. Consequently this light, when examined only by the analyzing plate B, shews no sign of polarization (97), (99), &c. This is experimentally true. But if elliptically polarized light is resolved in the same way, though neither of the resolved parts ever vanishes, yet their magnitudes vary: and therefore when examined with the analyzing plate it will appear to be partially polarized. This is also true.

141. Between two kinds of circularly polarized light there is an important distinction which we have not yet pointed out. We have seen that if $\alpha = 45°$ in Fresnel's rhomb, the light is circularly polarized: it is also circularly polarized if $\alpha = -45°$. For in the latter case

$$y = -a\sqrt{\tfrac{1}{2}}\cos\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

$$z = a\sqrt{\tfrac{1}{2}}\sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

and therefore
$$y^2 + z^2 = \frac{a^2}{2}.$$

The difference consists in the difference of direction of each particle's motion. In the former case,

$$\frac{z}{y} = \tan\left\{\frac{2\pi}{\lambda}(vt - x)\right\};$$

in the latter

$$\frac{z}{y} = -\tan\left\{\frac{2\pi}{\lambda}(vt - x)\right\}.$$

If we suppose the plane of reflection vertical, y measured upwards, and z to the right hand (looking in the direction of the wave's motion), then in the former case the revolution will be in the same direction as that of the hands of a watch, and in the latter case in the opposite direction. In the former case the expressions shew that the particles which were originally in a straight line will be at any time arranged as in a left-handed spiral; in the latter case, as in a right-handed spiral. A similar distinction exists between two kinds of elliptical polarization.

142. One of the most remarkable proofs of the correctness of the theory is this; if a second Fresnel's rhomb be placed to receive the light coming from the first, and if its position be similar, the emergent light is plane-polarized, but the new plane of polarization is inclined 2α to the former plane of polarization. The theoretical explanation is this; the vibrations in the plane of incidence are accelerated 90^0 by the first rhomb and 90^0 again by the second rhomb, more than those perpendicular to the plane of incidence. Consequently {taking up the investigation of (137)}, the vibration perpendicular to the plane of incidence being

$$a\cos\alpha\,.\sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

that parallel to the plane will be

$$a\sin\alpha\,.\sin\left\{\frac{2\pi}{\lambda}(vt - x) + 180^0\right\},$$

$$\text{or } -a\sin\alpha\,.\sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\}.$$

As these are always in the same proportion, the vibration is entirely in one plane, or the light is plane-polarized. But as the tangent of the angle made with the plane of reflection is $-\tan\alpha$, instead of $\tan\alpha$, which was its value before inci-

dence, the plane of polarization is inclined on the side of the plane of reflection opposite to that on which it was before, and by the same angle; the change of position therefore is 2α.

If the second rhomb is placed in a position 90^0 different from that of the first, the emergent light is similar to the incident light. For, the vibrations which were most accelerated by the first rhomb are least accelerated by the second rhomb, and *vice versâ*, so that the relation of their phases is not altered.

143. It is remarkable that in the only other case of reflection unaccompanied by refraction whose laws are well known to us, namely reflection at the surfaces of metals, the reflected ray appears to possess properties similar to those of light totally reflected within glass: if the incident light is plane-polarized, the reflected light is in fact elliptically polarized, and the difference of the phases varies with the angle of incidence. It is not however certain that we can refer the physical explanation to the same principles: all that we can seem able to conclude from it is that the reflection from metallic surfaces is not strictly analogous to the reflection of sound from a wall, but has a closer relation to reflection from a surface terminating a dense medium*. Whether any ex-

* It appears from Sir David Brewster's experiments that, in reflection from metals, the proportion of the vibrations parallel to the plane of reflection in the reflected ray to those in the incident ray, is less than the proportion of the vibrations perpendicular to the plane of reflection in the reflected ray to those in the incident ray. Consequently, after a great number of reflections from metallic surfaces, the reflections being all performed in the same plane, the vibrations parallel to that plane are diminished in a rapidly-decreasing geometrical series, and are soon insensible, and therefore the light appears to be polarized in the plane of reflection. This happens with a much smaller number of reflections from steel than from silver. It appears also that the alteration of phases of the two sets of vibrations is somewhat different for different metals, and different at different angles of incidence; beginning to be sensible at the incidence 40^0 nearly, and amounting at its maximum (which it has at an incidence of nearly 70^0) to about 90^0; perhaps to more, but Sir D. Brewster's statements leave it doubtful. It is also doubtful whether the phase of the vibrations parallel to the plane of incidence is accelerated or retarded. See Brewster on Elliptic Polarization, *Phil. Trans.* 1830. The nature of elliptically-polarized light had been sufficiently indicated in a few words by Fresnel, and the merit of this valuable paper consists entirely in shewing that the reflection of polarized light at a metallic surface produces light of that kind. The result of these laws, on the principles of the text, may be thus stated. If the incident vibration perpendicular to the plane of incidence is $a \cdot \sin(vt - x + A)$; the multiplier of the

planation could be founded on the supposition that the ether is absolutely terminated at the reflecting surface, or that the ether within the metal is in a rarer state than that external to it, is a point that has not been examined.

ON THE COLOURED RINGS PRODUCED BY INTERPOSING A CRYSTALLINE BODY BETWEEN A POLARIZING PLATE AND AN ANALYZING PLATE.

144. In (97) we have mentioned as one of the fundamental facts of polarization that if the planes of reflection of A and B in fig. 24 are at right angles to each other, the angles of incidence at both being the polarizing angles, the light reflected from A is incapable of being again reflected from B. If the eye be placed near B so as to observe the image of A, a very dark spot is seen at its center, and the whole image, though not quite so black as the central spot, is very obscure.

145. Now if we interpose between A and B a plate which possesses double refraction, the image of A is generally seen bright, but sometimes crossed by one or more dark brushes, and sometimes by rings of circular or more complicated figure, richly coloured. On inclining the plane of the interposed plate, the rings generally shift their places and are succeeded by others; shewing that the peculiar arrangement of colours and brushes depends on the relation of the direction of the rays to some fixed lines in the interposed plate. There are few substances which when interposed present exactly the same phænomena, but nearly all exhibit appearances of the same general character: gorgeous colours, arrayed in symmetrical forms, generally shifting with every change in the position of the interposed plate, and always altering as B

coefficient in the reflected ray p: the corresponding quantities for the other vibration $b \,.\, \sin(vt-x+B)$ and q: and the acceleration of the latter δ: then after n reflections the vibration perpendicular to the plane of incidence is

$$a \,.\, p^n \,.\, \sin(vt-x+A),$$

and that parallel to the plane of incidence

$$b \,.\, q^n \,.\, \sin(vt-x+B+n\delta).$$

is turned round its spindle. This class of phænomena is far the most splendid in Optics.

146. The interposition of a piece of common glass produces no effect. And even a doubly refracting substance produces no effect, if it be placed to receive the light either before it is polarized at A, or after it is analyzed at B. It seems therefore that a doubly refracting substance has generally the power of altering polarized light, in such a manner that the light, either from losing the character of polarization, or from a change in the plane of polarization, acquires according to certain complicated laws the capability of reflection. It appears however that it exerts no influence on common light which makes it incapable of polarization as usual, and that it does not alter polarized light so as to produce any alteration in the impression made on the eye unless it is subsequently analyzed.

Prop. 30. To explain generally the origin of the coloured rings.

147. The general explanation may be given thus. From experience*, as well from the theory of (106) &c., it appears that, whatever be the nature of light incident on a doubly refracting crystal, the two rays which it produces are polarized, one in one plane, and the other in the plane perpendicular to the former. That is, the vibrations of the incident ray are resolved into two sets, one in one direction and the other in the direction perpendicular to that, which produce waves that describe different paths; and one of these forms the Ordinary ray and the other the Extraordinary ray. And from the theory of (108) it appears that these two sets of waves will pass through the crystal with different velocities, and therefore on coming out of the crystal will be in different phases. Their union therefore will produce a kind of light not necessarily plane-polarized, or not necessarily polarized in the same plane as before passing through the crystal: and therefore their capability of reflection at the

* Quartz is the only well-established exception to this rule. It appears that neither the ordinary nor the extraordinary ray of quartz is strictly plane-polarized.

analyzing plate is, generally, restored. But, as the positions of the two planes of polarization, as well as the difference of velocity of the two rays, will depend upon the direction of the paths through the crystal, the nature of the light produced by the union of the two emergent streams will vary as the directions vary; and consequently the intensity of the light coming to the eye after analyzation will vary with the direction of the ray. Thus bright patches or curves of different intensity will be seen. The difference of phases, it may be easily conceived, is generally a function of λ, and thus the form or size of the curves may be different for differently coloured light. From the mixture of these differently sized and differently coloured curves, curves will be produced in which the mixture of colours is different at almost every point, as in the fringes of interference and in Newton's rings.

148. We have supposed here that neither plane of polarization of the rays in the crystal coincides with the plane of polarization of the light reflected from A. But conceive that in one direction of the ray, the plane of polarization of the ordinary ray coincides with the plane of polarization of light reflected from A. In that case the light reflected from A will produce in the crystal only the ordinary ray, (92) and (95), and consequently the crystalline separation of the rays is of no consequence, because only one of the rays exists. The ordinary ray emerges therefore from the crystal just as it entered, unmixed with any other ray, and therefore falls upon B in the same state as if it had not passed through the crystalline plate, and therefore, is not reflected. The same would be true, *mutatis mutandis*, if for another direction of the ray the plane of polarization of the extraordinary ray in the crystal coincided with the plane of polarization of light reflected from A. Thus if we determine all the directions of rays in which the plane of polarization of either the ordinary or the extraordinary ray coincides with the plane of reflection from A, the rays passing in those directions will not be capable of reflection from B, and the appearance presented to the eye by the rays passing in all these directions will be that of one or more black

lines not necessarily straight, cutting the coloured curves before mentioned.

149. If B be turned round its spindle till its plane of reflection coincides with that of A, the positions determined by the conditions of (148) will define the directions in which the light is most highly susceptible of reflection from B, and therefore one or more bright lines will be seen cutting the curves. If B be turned to any intermediate position, it will be found in the same way that the directions of rays, which make the plane of polarization of either the ordinary or the extraordinary ray to coincide with the plane of reflection either at A or at B, determine the form of lines which cut all the rings, and in which the intensity of light is uniformly the same as if the crystal were not interposed.

These particular cases are pointed out merely as matters of interest in the general explanation. The determination of the form of the uncoloured curves will be included in the general investigation of the intensity of light reflected in all directions from B.

Prop. 31. A plate of Iceland spar (or other uniaxal crystal, except quartz) is bounded by planes perpendicular to the axis of the crystal: light is incident nearly in the direction of the axis; to find the position of the front, and the velocity perpendicular to the front, of the ordinary and extraordinary waves: and the retardation of each produced by passing through the plate.

150. First, for the extraordinary ray. In fig. 34 let AB be the normal to the front of the incident wave, or the direction of the incident ray: BC the normal to the front of the extraordinary wave, which is not generally the same as the direction of the extraordinary ray: CD the direction of emergence parallel to AB: i the angle of incidence made by AB, i' the angle of refraction made by BC: v the velocity before incidence, v' the velocity of the extraordinary wave perpendicular to its front: T the thickness of the plate. The time of describing BC is $\dfrac{T}{v' \cos i'}$; the space which the wave would

in the same time have described in air is $\dfrac{Tv}{v' \cos i'}$. But since the front of the wave at incidence was perpendicular to AB at B, and at emergence perpendicular to CD at C, the whole space which the wave really has advanced is

$$BE = \frac{T \cos (i - i')}{\cos i'},$$

and therefore it has been retarded by a space in air equal to

$$\frac{T}{\cos i'} \left(\frac{v}{v'} - \cos i \,.\, \cos i' - \sin i \,.\, \sin i' \right).$$

151. Now $\sin i' = \dfrac{v'}{v} \sin i$. For if GH be a position of the front before incidence and BK after entrance, GB and HK must have been described in the same time; and therefore $GB : HK$ (or $\sin i : \sin i'$) :: velocity of incident wave : velocity of extraordinary wave perpendicular to its surface :: $v : v'$. And as the perpendicular to the refracting surface coincides with the axis of the crystal, we have by (112)

$$v' = \sqrt{(a^2 \cos^2 i' + c^2 \sin^2 i')}.$$

From these equations we find

$$\sin i' = \frac{a \sin i}{\sqrt{\{v^2 - (c^2 - a^2) \sin^2 i\}}},$$

$$\cos i' = \frac{\sqrt{(v^2 - c^2 \sin^2 i)}}{\sqrt{\{v^2 - (c^2 - a^2) \sin^2 i\}}},$$

$$v' = \frac{av}{\sqrt{\{v^2 - (c^2 - a^2) \sin^2 i\}}}.$$

Substituting, the retardation

$$= T \left\{ \frac{\sqrt{(v^2 - c^2 \sin^2 i)}}{a} - \cos i \right\}.$$

152. Next for the ordinary ray. This may be deduced from the last by putting a for c. For the expression in (112)

is changed to that in (111) by this alteration. Consequently the retardation for the ordinary ray is

$$T\left\{\frac{\sqrt{(v^2 - a^2 \sin^2 i)}}{a} - \cos i\right\}.$$

153. The only quantity that concerns us is the excess* of the latter above the former. Its value is

$$\frac{T}{a}\{\sqrt{(v^2 - a^2 \sin^2 i)} - \sqrt{(v^2 - c^2 \sin^2 i)}\}.$$

When i is small, this is nearly

$$= T\frac{c^2 - a^2}{2av} \sin^2 i.$$

This we shall call I.

154. In estimating then the displacement in the ether produced by these two separate pencils after emergence from the plate of crystal, if we represent that which is produced by the ordinary ray by a multiple of

$$\sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

we must represent that produced by the extraordinary ray by a multiple of

$$\sin\left[\frac{2\pi}{\lambda}\{vt - (x - I)\}\right],$$

$$\text{or}\quad \sin\left\{\frac{2\pi}{\lambda}(vt - x) + \frac{2\pi I}{\lambda}\right\}.$$

For, the extraordinary ray is less retarded than the ordinary ray by the space I in air, and therefore the displacement really caused by the extraordinary ray will correspond to that

* c is greater than a for Iceland spar, beryl, and all the crystals termed by some writers *negative*; and less than a for some varieties of apophyllite, and all crystals of their *positive* class.

which would have been produced at a space less advanced by I* if they had been equally retarded.

PROP. 32. A plate of Iceland spar &c. bounded by planes perpendicular to the axis of the crystal (as in Art. 150) is placed between the polarizing and analyzing plates, fig. 24: to investigate the intensity of the light in various parts of the image seen after reflection at B.

155. In fig. 35 conceive the direction of any ray to be perpendicular to the paper: let the plane passing through this ray and through the axis of the crystal {which in (111) we have termed the *principal plane* for that ray} make with the plane of first polarization the angle ϕ: and let the plane of polarization at the analyzing plate (which we shall call the *plane of analyzation*) make with the plane of first polarization the angle α. Let the vibration in the rays as first polarized be represented by

$$a \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

perpendicular to the plane of first polarization. On entering the crystal this is resolved into

$$a \cos \phi \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

perpendicular to the principal plane (which produces the Ordinary ray), and

$$a \sin \phi \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

parallel to the principal plane (which produces the Extraordinary ray). The former of these expressions may be assumed to be true after the Ordinary ray has emerged from the crystal, provided that we make the proper alteration in

* The same expression applies when two plates cut in the same way from crystals either of the same or of different kinds are applied together, I being now the space by which in the combination the extraordinary ray is less retarded than the ordinary ray. If in both plates the extraordinary ray is less retarded than the ordinary ray, or in both more, the effect of the combination is that of a thick plate: if in one it is less and in the other more retarded, the effect is that of a thin plate.

the value of x or t: but then for the Extraordinary ray we must, by (154), take the expression

$$a \sin \phi \,.\, \sin \left\{ \frac{2\pi}{\lambda}(vt - x) + \frac{2\pi I}{\lambda} \right\}.$$

If the rays entered the eye in this state, there would be no variation of intensity in the light coming in different directions through the crystal. For the intensity of the ordinary wave

$$= a^2 \cos^2 \phi,$$

and that of the extraordinary wave

$$= a^2 \sin^2 \phi,$$

the sum of which, or a^2, represents the intensity of the united waves (102), and this is constant. Now the analyzing plate being applied, those resolved parts only of the vibrations are preserved which are perpendicular to the plane of analyzation. That* furnished by the ordinary ray is

$$a \cos \phi \,.\, \cos(\phi + \alpha) \,.\, \sin \left\{ \frac{2\pi}{\lambda}(vt - x) \right\}:$$

and that furnished by the extraordinary ray is

$$a \sin \phi \,.\, \sin(\phi + \alpha) \,.\, \sin \left\{ \frac{2\pi}{\lambda}(vt - x) + \frac{2\pi I}{\lambda} \right\}.$$

The sum of these represents the displacement produced by the wave that enters the eye. Adding them, and expanding

$$\sin \left\{ \frac{2\pi}{\lambda}(vt - x) + \frac{2\pi I}{\lambda} \right\},$$

we find for the coefficient of $\sin \left\{ \frac{2\pi}{\lambda}(vt - x) \right\}$,

$$a \cos \phi \,.\, \cos(\phi + \alpha) + a \sin \phi \,.\, \sin(\phi + \alpha) \,.\, \cos \frac{2\pi I}{\lambda},$$

* As the analyzing plate does not transmit to the eye the whole of the vibrations perpendicular to its plane of polarization, we ought in strictness to multiply these expressions, in this and similar investigations, by a constant. The omission is of no consequence in comparing the intensities of different parts of the image.

and for the coefficient of $\cos\left\{\frac{2\pi}{\lambda}(vt-x)\right\}$,

$$a\sin\phi\,.\,\sin(\phi+\alpha)\,.\,\sin\frac{2\pi I}{\lambda}.$$

The sum of the squares of these coefficients is to be taken for the measure of the intensity {as in (17) and (23)}. This sum is

$$a^2\cos^2.\,\phi\,.\,\cos^2(\phi+\alpha)+a^2\sin^2\phi\,.\,\sin^2(\phi+\alpha)$$

$$+2a^2\sin\phi\,.\,\cos\phi\,.\,\sin(\phi+\alpha)\,.\,\cos(\phi+\alpha)\,.\,\cos\frac{2\pi I}{\lambda},$$

$$\text{or }\frac{a^2}{2}\left\{1+\cos 2\phi\,.\,\cos(2\phi+2\alpha)\right.$$

$$\left.+\sin 2\phi\,.\,\sin(2\phi+2\alpha)\,.\,\cos\frac{2\pi I}{\lambda}\right\},$$

$$\text{or }a^2\left\{\cos^2\alpha-\sin 2\phi\,.\,\sin(2\phi+2\alpha)\,.\,\sin^2\frac{\pi I}{\lambda}\right\}.$$

156. This gives the intensity of the light that enters the eye in a given direction, or the brightness of one point of the visible image. To determine what point of the image it is, we have only to remark that this ray makes the angle i with the ray that passes in the direction of the axis, in a plane that is inclined $\phi+\alpha$ to the plane of analyzation (supposing that we look in the direction of the ray's motion), measuring from the top to the right. By the reflection at the analyzing plate, this course of the rays is inverted with regard to *up* and *down*, while it is not altered with regard to *right* and *left:* but then, as the eye is placed to receive the light in the direction opposite to that in which we look in studying figure 35, there is another inversion with regard to *right* and *left*, but none with regard to *up* and *down*. On the whole therefore, this ray comes from a point whose apparent angular distance from a certain point through which the rays pass parallel to the axis is i, which distance is measured in a direction that makes the angle $\phi+\alpha$ or ψ with the plane of analyzation, measuring from the upper part of the plane to the right. If a plate of tourmaline or a Nicol's prism were used for analyzing

plate, the angular measure would be made from the upper part to the left. In the image presented to the eye, i may be considered as a radius vector, and ψ the angle that it makes with the upper part of the line that represents the plane of analyzation. The brightness, putting ψ for $\phi + \alpha$, is

$$a^2 \left\{\cos^2 \alpha - \sin (2\psi - 2\alpha) \,.\, \sin 2\psi \,.\, \sin^2 \frac{\pi I}{\lambda}\right\}.$$

157. Let $\alpha = 90^\circ$, or let the analyzing plane be in the position in which no light is reflected without the interposition of the crystal. The expression becomes

$$a^2 \sin^2 2\psi \,.\, \sin^2 \frac{\pi I}{\lambda}.$$

This is 0, whatever be the value of λ and of I, when

$$\sin^2 2\psi = 0:$$

that is, when

$$\psi = 0, \quad \text{or} = 90^\circ, \quad \text{or} = 180^\circ, \quad \text{or} = 270^\circ.$$

This shews that, whatever be the colour of the incident light, there is a black cross, passing through that point of the image which is formed by the light that is parallel to the axis. For all intermediate values of ϕ it vanishes only when

$$\frac{\pi I}{\lambda} = 0, \ \pi, \ 2\pi, \ \&c., \quad \text{or } I = 0, \ \lambda, \ 2\lambda, \ \&c.,$$

or

$$\sin i = 0, \ \sqrt{\frac{2av\lambda}{(c^2 - a^2)\,T}}, \ \sqrt{\frac{4av\lambda}{(c^2 - a^2)\,T}}, \ \sqrt{\frac{6av\lambda}{(c^2 - a^2)\,T}}, \ \&c.,$$

and the light is brightest and $= a^2 \sin^2 2\psi$ when

$$I = \frac{\lambda}{2}, \ \frac{3\lambda}{2}, \ \&c.,$$

$$\text{or } \sin i = \sqrt{\frac{av\lambda}{(c^2 - a^2)\,T}}, \ \sqrt{\frac{3av\lambda}{(c^2 - a^2)\,T}}, \ \&c.$$

The four spaces between the arms of the cross are therefore occupied by bright and dark rings, the radii of the bright rings being as $\sqrt{1}$, $\sqrt{3}$, $\sqrt{5}$, &c. and those of the dark rings as $\sqrt{2}$, $\sqrt{4}$, $\sqrt{6}$, &c. The radii are inversely as $\sqrt{T}$, and the rings are therefore smaller with a thick plate than with a thin one. The radii are also inversely as $\sqrt{\frac{c^2-a^2}{av}}$ *; and as this expression may conveniently be taken as a measure of the doubly refracting power of the crystal, the rings are less with a powerful doubly refracting crystal than with one which has that property in a feeble degree. The radii vary also as $\sqrt{\lambda}$ (considering $\frac{av}{c^2-a^2}$ as independent of λ), and thus are larger for red light than for blue light. This produces exactly the same effect which we have noticed in speaking of interference fringes and Newton's rings, (51), (66), and (72): the rings which at first are black and white have very soon a mixture of colours, different at every successive ring, and finally disappear from the mixture of all in almost equal proportions. If $\frac{av}{c^2-a^2}$ were constant, the proportion of the radii for different colours, and consequently the mixture of colours, would be nearly the same as in Newton's rings. But $\frac{av}{c^2-a^2}$ is generally a function of λ: the radii of rings of different colours vary therefore as $\sqrt{\frac{av\lambda}{c^2-a^2}}$, and the colours are not the same as those of Newton's scale. To such an extent, and so differently in different crystals, does $\frac{av}{c^2-a^2}$ vary with λ, that in one variety of the uniaxal apophyllite Sir John Herschel found that $\frac{av\lambda}{c^2-a^2}$ was almost exactly constant, so that more than 35 rings were visible: while in another variety c^2-a^2 was positive for the rays from

* The reader must not infer from this expression that there are no rings when c^2 is less than a^2. On going through the whole of the investigation it will be seen that the very same expressions will apply, putting only a^2-c^2 instead of c^2-a^2.

one end of the spectrum and negative for those from the other end, and $=0$ for the intermediate rays, and only one or two rings were visible.

158. Let $\alpha = 0^{\circ}$, or the plane of reflection at the analyzing plate, coincide with that at the polarizing plate. The expression for the intensity is

$$a^2\left(1 - \sin^2 2\psi \, . \sin^2 \frac{\pi I}{\lambda}\right).$$

This expression, added to that discussed in (157), produces a sum a^2. Consequently the intensity at any point of the image in this case is complementary to that in the case of (157). Thus, instead of a black cross interrupting the rings, there is a bright cross interrupting the rings: instead of the dark rings having radii

$$\sqrt{\frac{2av\lambda}{(c^2-a^2)\,T}}, \quad \sqrt{\frac{4av\lambda}{(c^2-a^2)\,T}}, \quad \&c.,$$

and the bright rings having the radii

$$\sqrt{\frac{av\lambda}{(c^2-a^2)\,T}}, \quad \sqrt{\frac{3av\lambda}{(c^2-a^2)\,T}}, \quad \&c.,$$

the bright rings have the former and the dark ones the latter.

159. In the general case, there is no variation of the intensity with different values of I or i, (that is, there is a brush of some sort with light of uniform intensity throughout, interrupting the rings), when

$$\sin(2\psi - 2\alpha) \, . \sin 2\psi = 0.$$

For, the succession of rings depends on the alteration of values of $\sin^2 \dfrac{\pi I}{\lambda}$: and this is removed by the evanescence of its multiplier. This equation gives

$$\psi = 0, \quad \text{or} = 90^{\circ}, \quad \text{or} = 180^{\circ}, \quad \text{or} = 270^{\circ}, \quad \text{or} = \alpha, \quad \text{or} = 90^{\circ} + \alpha,$$
$$\text{or} = 180^{\circ} + \alpha, \quad \text{or} = 270^{\circ} + \alpha.$$

Consequently there are two rectangular crosses, inclined α to each other, which interrupt the rings. If a Nicol's prism is used, it is easily seen that one of these crosses has respect to the plane of polarization of the polarizing plate, and the other to that of the Nicol's prism. The intensity of the light in these crosses is $a^2 \cos^2 \alpha$. For the parts between $\psi = 0$, $\psi = \alpha$, or between $\psi = 90^0$, $\psi = 90^0 + \alpha$, or between $\psi = 180^0$, $\psi = 180^0 + \alpha$, or between $\psi = 270^0$, $\psi = 270^0 + \alpha$, the multiplier of $\sin^2 \frac{\pi I}{\lambda}$ is positive, and the light is therefore greatest when $I = \frac{\lambda}{2}, = \frac{3\lambda}{2}$, &c., and least when $I = \lambda, = 2\lambda$, &c.: these four sectors are therefore occupied by portions of rings nearly similar to those in (157), the intensity for the portions of the bright rings being

$$\frac{a^2}{2}\{1 + \cos(4\psi - 2\alpha)\}, \text{ or } a^2 \cos^2(2\psi - \alpha),$$

and that for the portions of the darker rings $a^2 \cos^2 \alpha$.

But for the parts between $\psi = \alpha$, $\psi = 90^0$, &c., the multiplier of $\sin^2 \frac{\pi I}{\lambda}$ is negative: the light is least when $I = \frac{\lambda}{2}$, $\frac{3\lambda}{2}$, &c., and greatest when $I = \lambda, 2\lambda$, &c.: these sectors therefore are occupied by portions of rings nearly similar to those in (158), the intensity of the portions of the bright rings being $a^2 \cos^2 \alpha$, and that of the fainter rings

$$a^2 \cos^2(2\psi - \alpha).$$

The brighter rings in the last-mentioned sectors have the same radii and the same brightness as the darker rings in those first mentioned: and this brightness is the same as the brightness in the eight rays of the crosses.

PROP. 33. In the last experiment, Fresnel's rhomb is placed between the polarizing plate and the plate of crystal, with its plane of reflection inclined 45^0 to the plane of polarization, so that the light incident on the crystal is circularly

polarized: to find the intensity of the light after reflection from B, and the form of the coloured rings.

160. Resolve the vibration

$$a \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

which is perpendicular to the plane of first polarization,

$$\text{into } \frac{a}{\sqrt{2}} \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\},$$

perpendicular to the plane of reflection in the rhomb

$$\text{and } \frac{a}{\sqrt{2}} \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

parallel to that plane. The latter of these, by (136), has its phase increased by 90°, and therefore on coming out of the rhomb the vibrations may be represented by

$$\frac{a}{\sqrt{2}} \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

perpendicular to the plane of reflection

$$\text{and } \frac{a}{\sqrt{2}} \cos\left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

parallel to that plane. Resolving these in directions perpendicular and parallel to the principal plane of the crystal, we find;

Vibration which produces Ordinary ray

$$= \frac{a}{\sqrt{2}} \cos(45^\circ - \phi) \,.\, \sin\left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

$$+ \frac{a}{\sqrt{2}} \sin(45^\circ - \phi) \,.\, \cos\left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

$$= \frac{a}{\sqrt{2}} \sin\left\{\frac{2\pi}{\lambda}(vt - x) + 45^\circ - \phi\right\}.$$

Vibration which produces Extraordinary ray

$$= -\frac{a}{\sqrt{2}} \sin (45^\circ - \phi) \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

$$+ \frac{a}{\sqrt{2}} \cos (45^\circ - \phi) \,.\, \cos \left\{\frac{2\pi}{\lambda}(vt - x)\right\}$$

$$= \frac{a}{\sqrt{2}} \cos \left\{\frac{2\pi}{\lambda}(vt - x) + 45^\circ - \phi\right\}.$$

On emerging from the crystal, the Ordinary vibration being represented by the same expression, the Extraordinary vibration must be represented by

$$\frac{a}{\sqrt{2}} \cos \left\{\frac{2\pi}{\lambda}(vt - x) + 45^\circ - \phi + \frac{2\pi I}{\lambda}\right\}.$$

The resolved parts of these perpendicular to the plane of analyzation (which are the only parts that reach the eye) are

$$\frac{a}{\sqrt{2}} \cos (\alpha + \phi) \,.\, \sin \left\{\frac{2\pi}{\lambda}(vt - x) + 45^\circ - \phi\right\}$$

$$+ \frac{a}{\sqrt{2}} \sin (\alpha + \phi) \,.\, \cos \left\{\frac{2\pi}{\lambda}(vt - x) + 45^\circ - \phi + \frac{2\pi I}{\lambda}\right\}.$$

Expanding the last term, the coefficients of

$$\sin \left\{\frac{2\pi}{\lambda}(vt - x) + 45^\circ - \phi\right\}$$

$$\text{and } \cos \left\{\frac{2\pi}{\lambda}(vt - x) + 45^\circ - \phi\right\},$$

are

$$\frac{a}{\sqrt{2}} \cos (\alpha + \phi) - \frac{a}{\sqrt{2}} \sin (\alpha + \phi) \,.\, \sin \frac{2\pi I}{\lambda},$$

$$\text{and } \frac{a}{\sqrt{2}} \sin (\alpha + \phi) \,.\, \cos \frac{2\pi I}{\lambda},$$

and the intensity of the light, or the sum of the squares, is

$$\frac{a^2}{2}\left\{1 - \sin (2\alpha + 2\phi) \,.\, \sin \frac{2\pi I}{\lambda}\right\},$$

$$\text{or } \frac{a^2}{2}\left\{1 - \sin 2\psi \,.\, \sin\frac{2\pi I}{\lambda}\right\}.$$

161. Since α does not enter into this expression, the appearance will not be altered on turning B round its spindle. When $\sin 2\psi = 0$, that is when $\psi = 0$, or 90°, or 180°, or 270°, the intensity is $\frac{a^2}{2}$: this shews that there is a cross with light of mean intensity interrupting the rings. When

$$\psi \text{ is } > 0 < 90^\circ, \text{ or } > 180^\circ < 270^\circ,$$

the expression is maximum when

$$\frac{2\pi I}{\lambda} = \frac{3\pi}{2},\ \frac{7\pi}{2},\ \&c.$$

and minimum when

$$\frac{2\pi I}{\lambda} = \frac{\pi}{2},\ \frac{5\pi}{2},\ \&c.$$

or maximum when

$$I = \frac{3\lambda}{4},\ \frac{7\lambda}{4},\ \&c.$$

and minimum when

$$I = \frac{\lambda}{4},\ \frac{5\lambda}{4},\ \&c.$$

When ψ is $> 90^\circ < 180^\circ$ or $> 270^\circ < 360^\circ$, the expression is maximum when

$$I = \frac{\lambda}{4},\ \frac{5\lambda}{4},\ \&c.$$

and minimum when

$$I = \frac{3\lambda}{4},\ \frac{7\lambda}{4},\ \&c.$$

Thus it appears that of the four quadrants into which the cross divides the image, each opposite pair is similar, but each adjacent pair is dissimilar: the bright rings in one quadrant having the same radii as the dark rings in the next quadrant. And on comparing these expressions with those in (157), it will be seen that the effect of placing Fresnel's

rhomb has been to push the rings outward by $\frac{1}{4}$ of an order in two opposite quadrants, and to pull them in by $\frac{1}{4}$ of an order in the other two opposite quadrants. At the same time the cross which was perfectly black has now some light. The most important difference of character however which the use of Fresnel's rhomb produces is the unchangeability of appearances as B is turned round.

If we compared the rings produced with the same position of Fresnel's rhomb by two crystals, in one of which c^2 was $> a^2$ and in the other of which c^2 was $< a^2$, then for a given order of rings, that is for those in which the magnitude of $\frac{I}{\lambda}$, without respect to its sign, is the same, $\sin \frac{2\pi I}{\lambda}$ would be positive for the first and negative for the second, or *vice versâ*. Consequently the bright rings of one crystal would correspond to the dark ones of the other. But we have seen that the bright rings of one quadrant correspond to the dark rings of the neighbouring quadrant. Consequently the rings presented by one of the crystals would be the same as those presented by the other, supposing the latter rings turned round 90^0. This affords a convenient method of determining whether the double refraction of a uniaxal crystal is of the same kind as that of a standard crystal (for instance Iceland spar) or of the opposite kind.

PROP. 34. A plate of a biaxal crystal whose optic axes make a small angle with each other (as nitre or arragonite) is bounded by planes perpendicular to the plane passing through the axes and nearly perpendicular to each axis; light is incident at a small angle of incidence: to find the difference of retardation of the two rays.

162. The accurate solution of this problem leads to some rather complicated expressions: and we shall therefore content ourselves with a very approximate solution analogous to that found in (153). We have found there that the difference of retardation was nearly

$$= T\frac{c^2 - a^2}{2av}\sin^2 i, \text{ or nearly } = T\frac{v\,(c^2 - a^2)}{2a^3}\sin^2 i,$$

where the difference of the squares of velocities of the two waves was $(c^2-a^2)\sin^2 i'$: or that the difference of retardation was nearly $=\frac{Tv}{2a^3}\times$ difference of squares of velocities of the two waves. As the difference of retardation arises solely from the difference of velocities, we shall suppose the same proportion to be true here. Now by (125) neither of the rays undergoes Ordinary refraction, or has a constant velocity. Still, even in extreme cases, the velocity of one is so nearly a constant $= a$, that in a calculation depending almost wholly on the difference there will be no sensible error in considering one as constant and $= a$. And by (123), putting v' for the velocity of the other,

$$a^2 - v'^2 = Ca^4 \,.\, \sin m' \,.\, \sin n',$$

where m' and n' are the angles made by the normal to its front with the two optic axes of the crystal (C being always small).

Hence the difference of retardations

$$= \frac{TCav}{2} \sin m' \,.\, \sin n'.$$

Now let us consider the system of rays in air which on entering the crystal will pass in the directions that we have described. Let m and n be the angles made by the same ray in air with the rays which on entering the crystal will pass in the directions of the optic axes. As all the refracted rays (represented by the normals to the fronts of the waves) are in the same planes perpendicular to the refracting surface as the incident rays, and as all the angles of refraction are very nearly in the same proportion to the angles of incidence, it follows that all the other small angles depending on them, and their sines, are nearly in the same ratio.

Hence

$$\sin m' = \frac{a}{v} \sin m \text{ nearly}, \quad \text{and } \sin n' = \frac{a}{v} \sin n \text{ nearly}:$$

and therefore the difference of retardations is

$$\frac{TCa^3}{2v} \sin m \cdot \sin n.$$

This as before we shall call I.

PROP. 35. A plate cut from a biaxal crystal, as in (162), is placed between the polarizing and analyzing plates: to investigate the intensity of the light in different points of the image seen after reflection from B.

163. Let ϕ be taken now to represent the angle made by the plane of polarization of either ray with the plane of first polarization, and the expression of (155), which is founded on no supposition except that the planes of polarization of the two rays are perpendicular to each other, will apply to this case. The intensity of light is therefore

$$a^2 \left\{\cos^2 \alpha - \sin 2\phi \cdot \sin (2\phi + 2\alpha) \cdot \sin^2 \frac{\pi I}{\lambda}\right\}.$$

Conceive fig. 36 to be the projection of the directions of the rays and planes on a sphere (or rather on the tangent plane to a sphere) of which the eye is the centre and whose radius is r. Let A, B, thus represent the optic axes, P any ray under consideration, DE the plane of first polarization. Put β for the angle made by the plane passing through the optic axes of the crystal with the plane of first polarization. Let PQ bisect the angle APB: then, by (124), PQ represents the plane of polarization of one ray, and therefore

$$PQA = \phi + \beta.$$

$$\text{Also} \quad \sin m = \frac{AP}{r} \text{ nearly}, \qquad \sin n = \frac{BP}{r} \text{ nearly},$$

and therefore

$$I = \frac{TCa^3}{2vr^2} \cdot AP \cdot BP.$$

164. Now the form of the brushes interrupting the rings will be discovered by making the multiplier of $\sin^2 \frac{\pi I}{\lambda} = 0$.

This gives

$$\sin 2\phi = 0, \quad \text{or} \quad \sin(2\phi + 2\alpha) = 0.$$

Consequently

$$\tan(2\phi + 2\beta) = \tan 2\beta, \qquad \text{or} = \tan(2\beta - 2\alpha).$$

Now refer P to the point C bisecting AB, by rectangular coordinates, x being measured in the direction CA and y perpendicular to it; let $CA = b$. Then

$$\tan PAF = \frac{y}{x-b} : \quad \tan PBF = \frac{y}{x+b} :$$

whence $\tan(2\phi + 2\beta) = \tan 2PQA = \tan(PBF + PAF)$

(because PQ bisects the angle at P)

$$= \frac{2xy}{x^2 - b^2 - y^2}.$$

Hence the brushes are determined by these equations:

$$\frac{2xy}{x^2 - b^2 - y^2} = \tan 2\beta,$$

$$\text{or} \quad (x^2 - b^2 - y^2) \tan 2\beta - 2xy = 0;$$

$$\frac{2xy}{x^2 - b^2 - y^2} = \tan(2\beta - 2\alpha),$$

$$\text{or} \quad (x^2 - b^2 - y^2) \tan(2\beta - 2\alpha) - 2xy = 0.$$

These are evidently equations to hyperbolas, of which C is the center. As in both of them $y = 0$ when $x = \pm b$, the hyperbolas defined by both equations pass through A and B. The position of the asymptotes will be determined by supposing x and y very great compared with b: this gives in the first equation

$$\frac{y^2}{x^2} + 2\cot 2\beta \frac{y}{x} - 1 = 0, \quad \text{or} \quad \frac{y}{x} = +\tan\beta \ \text{or} \ -\cot\beta,$$

and similarly in the second

$$\frac{y}{x} = +\tan(\beta - \alpha) \ \text{or} \ -\cot(\beta - \alpha).$$

This shews that both hyperbolas are rectangular, and that the asymptotes of one are parallel and perpendicular to the plane of first polarization; and those of the other inclined to them by α, or (if Nicol's prism is used) parallel and perpendicular to the plane of polarization of the Nicol's prism. The intensity of light in the brushes is

$$a^2 \cos^2 \alpha.$$

165. When $\beta = 0$, or $= 90^0$, $\tan 2\beta = 0$, and the first hyperbolas are changed into two straight lines, one in the direction of FC, and the other perpendicular to it, passing through C. Similarly when $\beta = \alpha$, or $= 90^0 + \alpha$, the second hyperbolas are changed into a similar cross. Whatever be the value of β, if $\alpha = 0$ or $= 90^0$, the two pairs of hyperbolas coincide: but the value $\alpha = 0$ gives for the intensity a^2, or the brush is bright: and the value $\alpha = 90^0$ gives for the intensity 0, or the brush is black.

166. The nature of the rings is determined by the variations of value of the last term

$$-\sin 2\phi \,.\, \sin(2\phi + 2\alpha) \,.\, \sin^2 \frac{\pi I}{\lambda}.$$

$$\text{When } \phi > 0 < 90^0 - \alpha, \qquad \text{or } > 90^0 < 180^0 - \alpha,$$

$$\text{or } > 180^0 < 270^0 - \alpha, \qquad \text{or } > 270^0 < 360^0 - \alpha,$$

(the limits of which are determined by the hyperbolas already described) the brightness is greatest when

$$I = 0, \quad \text{or} = \lambda, \quad \text{or} = 2\lambda, \text{ \&c.}$$

and least when

$$I = \frac{\lambda}{2}, \frac{3\lambda}{2}, \text{ \&c.}$$

When $\phi > 90^0 - \alpha < 90^0$, the brightness is greatest when $I = \frac{\lambda}{2}$, $\frac{3\lambda}{2}$, &c. and least when $I = 0, \lambda, 2\lambda$, &c.

The cases when $\alpha = 0$ or $\alpha = 90^0$ will be very easily investigated by the reader.

167. It is plain then that the general appearance will be rings, interrupted by the brushes, in such a way that the bright rings on one side of the brushes correspond to the dark rings on the other side (except $\alpha = 0$ or $\alpha = 90^0$, when the number of brushes is diminished, and the rings on opposite sides correspond); and that the form of these rings will be determined by the equation $I = \lambda \times$ constant; or, by (163),

$$\frac{TCa^3}{2vr^2} AP \cdot BP = \lambda \times \text{constant},$$

$$\text{or } AP \cdot BP = \frac{2vr^2\lambda}{TCa^3} \times \text{constant},$$

where the constant determines the order of the rings. The curves determined by this equation are of the kind called *lemniscates*. If the constant is small, they will be nearly circles of which A and B are centers. As the constant is increased, the circles become elongated towards C: at last they become a single curve like the figure 8 crossing at C: then a single curve like a ring nipped so as almost to meet in the middle: and afterwards a ring slightly flattened. Combining this determination with those of (164) and (166) it is seen that, supposing values of β and α similar to those in figs. 35 and 36, the general appearance is that of fig. 37, where the curves represent the dark rings.

168. Since $AP \cdot BP \propto$ constant that determines the order, the radii of the successive rings, when they are small and nearly circular, are as 1, 2, 3, &c. nearly. In this respect they differ from those of a uniaxal crystal, which are as $\sqrt{1}, \sqrt{2}, \sqrt{3}$, &c. (157). And since $AP \cdot BP \propto \frac{1}{T}$, the rings are smaller with a thick plate of a given crystal than with a thin plate. And since $AP \cdot BP \propto \frac{1}{C}$, the rings are smaller with a crystal that produces a great difference in the velocity of the two rays than with one whose energy is feeble. And since $AP \cdot BP \propto \lambda$ or rather $\propto \frac{v\lambda}{Ca^3}$, the curves are *cæteris paribus* larger for red than for blue rays.

169. There is however one difference between the curves for the different colours which in its nature is unlike any thing else that we have yet seen. It is, that the optic axes for different colours do not coincide. In every instance however the alteration of place is symmetrical with regard to the two axes. Thus the two red axes may be less inclined than the two blue axes, or *vice versâ*, but the angle between one red and one blue axis is the same as that between the other red and the other blue. In one or two instances this angle amounts to nearly 10°. The consequence is that the colours are not the same in different parts of the rings of the same order. Suppose for instance (as in nitre) the red axes are less inclined than the blue. As the red rings are larger than the blue, we shall, on taking points exterior to A and B, find positions where all the colours are mixed or all are absent, and therefore the rings are nearly white and black. If we trace the same rings to the positions between A and B, the red rings will very much overshoot the blue rings, and therefore the rings have the colour peculiar perhaps to a high order in Newton's scale.

170. It was till very lately supposed that the axes of the different colours are all in the same plane. Sir J. Herschel has discovered that in some instances (in borax for example) this is not true: the planes, however, as far as yet observed, all pass through the line bisecting the angle formed by the two axes. The reader will have little difficulty in conjecturing the nature of the alteration which this irregularity produces in the colours of the curves.

Prop. 36. In the experiment of Prop. 35, Art. 163, Fresnel's rhomb is interposed between the polarizing plate and the crystal: to find the form, &c. of the coloured curves.

171. As in (160), the intensity of light at any point is

$$\frac{a^2}{2}\left\{1 - \sin(2\phi + 2\alpha) \, . \sin\frac{2\pi I}{\lambda}\right\}.$$

There is a brush interrupting the rings where

$$\sin(2\phi + 2\alpha) = 0:$$

this is the same equation as that which determines the second hyperbolas in (164), and which when $\beta = \alpha$ or $= 90^\circ + \alpha$ becomes a cross. When $\sin(2\phi + 2\alpha)$ is positive, the intensity is maximum if $I = \frac{3\lambda}{4}, \frac{7\lambda}{4}$, &c., and minimum if $I = \frac{\lambda}{4}$, $\frac{5\lambda}{4}$, &c.: and the contrary when $\sin(2\phi + 2\alpha)$ is negative. These spaces are separated by the brush: consequently the bright rings on one side of the brush correspond to the dark rings on the other side. The form of the rings is just the same as in (167).

PROP. 37. A plate of uniaxal or biaxal crystal, cut in any direction different from those of (150) and (162), is placed between the polarizing and analyzing plates: to find the appearance presented to the eye.

172. The general expression for the brightness in (155),

$$\frac{a^2}{2}\left\{1 + \cos 2\phi \,.\, \cos(2\phi + 2\alpha) + \sin 2\phi \,.\, \sin(2\phi + 2\alpha) \,.\, \cos\frac{2\pi I}{\lambda}\right\}$$

applies to this case. To confine ourselves to the most important instances we will make $\alpha = 90^\circ$, which reduces the expression to

$$\frac{a^2}{2}\sin^2 2\phi\left(1 - \cos\frac{2\pi I}{\lambda}\right).$$

By I is meant here the space that one ray (which whether in uniaxal or biaxal crystals we shall call the Ordinary ray) is retarded more than the Extraordinary ray, and to which the two expressions in (162) still apply, observing only that in the former i' is the angle made with the axis. The essential difference between this case and that of (155) and (162) is, that here I is large for all rays which pass nearly perpendicular to the plate:

(1) If the plate be thick, all traces of colours will disappear (as we have seen in several cases of interference where one ray had gained many multiples of λ on another). For, I being considerable, a very

small variation of λ will make $\frac{2\pi I}{\lambda}$ vary by 2π: and thus, for the various rays included in every small portion of the spectrum, $\cos\frac{2\pi I}{\lambda}$ will have all its values, positive and negative, and the sum of all these values will be $=0$. The intensity of light will therefore be $\frac{1}{2}\sin^2 2\phi\times$ incident light, a proportion which is the same for all the colours. On turning the plate round in its plane, ϕ will vary from 0 to 360°, and the light will disappear four times. It will be greatest when $\phi=45°$, 135°, &c.

(2) Colours may however be produced by crossing two plates of very nearly the same thickness cut in the same manner from the same crystal. For let I be the retardation of the Ordinary above the Extraordinary ray in the first, I' that in the second; I' will be very nearly equal to I. And, the plates being at right angles to each other, the Ordinary ray of the first will be the Extraordinary ray of the second. I' therefore will be the acceleration in the second plate of the same vibrations for which I was the retardation in the first: and therefore the whole retardation is $I-I'$, and the brightness is now

$$\frac{a^2}{2}\sin^2 2\phi\left\{1-\cos\frac{2\pi(I-I')}{\lambda}\right\},$$

$$\text{or } a^2\sin^2 2\phi\,.\,\sin^2\frac{\pi(I-I')}{\lambda}.$$

The space $I-I'$ may be so small that the arc $\frac{\pi(I-I')}{\lambda}$ may differ little (not more than a fraction of π or a small multiple of π) for differently coloured rays, and then there will be vivid colours.

(3) Colours may also be produced by applying together, with their axes parallel, two plates cut from uniaxal crystals, one of the positive and one of the negative class (as quartz and beryl). For in one of these the

Ordinary ray is most retarded, and in the other the Extraordinary ray is most retarded: and as the Ordinary ray in one forms the Ordinary ray in the other, the ray which is most retarded in the first is least retarded in the second, and thus the difference of retardations may be made as small as we please.

(4) From the bodies which crystallize in laminæ it is frequently possible to detach a plate so thin that it will exhibit colours: for instance sulphate of lime, or mica. Both these are biaxal: in the former the axes are in the plane of the laminæ: in the latter they are in a plane perpendicular to it, but widely separated.

(5) In all these cases, the colours do not form small rings, as in the cases that we have treated at length, but are diffused in broad sheets. This arises merely from the circumstance that the expression for I or $I-I'$ varies very slowly with the variation of incidence. In sulphate of lime, for instance, a ray perpendicular to the laminæ makes

$$m = 90^\circ, \qquad n = 90^\circ:$$

a ray inclined to this will produce very little alteration in $\sin m' . \sin n'$ on which I depends. The same is true in mica, where the ray makes equal angles with the two axes. If it be inclined in the plane of the axes, $\sin m' . \sin n'$ is diminished: if perpendicular to that plane, $\sin m' . \sin n'$ will be increased.

173. If the thickness of a lamina of sulphate of lime or mica is such that, for a ray perpendicular to the lamina, $I = \dfrac{\lambda}{4}$ for mean rays, the lamina may be used instead of Fresnel's rhomb. For here the light which is incident is resolved into two sets of vibrations at right angles to each other, and one of these is retarded in its phases by 90° more than the other; which is precisely the effect of Fresnel's rhomb. There is however this difference between them. In Fresnel's rhomb, whatever be the colour of the light, the retardation of the

phase is exactly 90°, or the corresponding retardation in space is exactly $\frac{\lambda}{4}$ whatever* the value of λ may be. In the crystallized plate, on the contrary, the retardation for mean rays is exactly $\frac{\lambda}{4}$, but it is greater than $\frac{\lambda}{4}$ for blue rays, and less than $\frac{\lambda}{4}$ for red rays. This is seen most distinctly on putting several such laminæ together, when the light which is reflected from the analyzing plate is coloured: whereas on putting together several Fresnel's rhombs, there is no such colour. It is plain that, in substituting such a lamina for Fresnel's rhomb, the plane of vibration of that ray which is least retarded corresponds to the plane of reflection in the rhomb.

If a thicker plate (for instance one that produces a difference of retardations amounting to 6λ for mean rays) be placed with its planes of polarization inclined 45° to that of first polarization, the effect on the rings of a uniaxal crystal is very remarkable. In two opposite quadrants, the ray which is most retarded furnishes the Ordinary ray, and in the other two the same furnishes the Extraordinary ray. In two opposite quadrants therefore the difference of the paths of the rays of the uniaxal crystal is increased, and in the others diminished; and therefore in two the colours are those belonging to distant rings, while in the other two the colours of the fifth or sixth ring are pure white and black, as if they were close to the center.

174. The investigations which we have given will apply to all the crystalline bodies whose laws of double refraction are accurately known. Quartz has been mentioned as an exception to the common laws of uniaxal crystals. It appears that the phænomena which it exhibits may be perfectly represented by supposing the Ordinary ray to consist of elliptically polarized light whose greater axis is perpendicular to the principal plane, and the Extraordinary ray to consist of ellip-

* This is not strictly true, as the same angle of incidence in the rhomb does not produce exactly the same effect for all rays: but it is much more exact than the other.

tically polarized light whose greater axis is in the principal plane: these two rays having also the difference* mentioned in (141): and the ellipses being changed to circles when the direction of the rays coincides with the axis of the crystal. It is also necessary to suppose that the axis of revolution of the spheroid (prolate for quartz) in which the Extraordinary wave is supposed to diverge (115) is less than the radius of the sphere into which the Ordinary wave diverges. For these investigations we must refer the reader to the *Cambridge Transactions*, Vol. IV.

PROP. 38. In every case where the interposed crystal resolves the light into two rays polarized in planes at right angles to each other, on turning the analyzing plate 90° the intensity of the light at each point is complementary to what it was before.

175. This is seen from the expression of (155). On putting $90^0 + \alpha$ for α, the expression becomes

$$\frac{a^2}{2}\left\{1-\cos 2\phi.\cos(2\phi+2\alpha)-\sin 2\phi.\sin(2\phi+2\alpha).\cos\frac{2\pi l}{\lambda}\right\},$$

which added to that in (155) makes a^2. Thus if in one case there is black, in the other there will be white: if in one there is an excess of red at any point and an absence of blue, in the other there will be an absence of red at the same point and an excess of blue, &c.

If instead of an analyzing plate we use the doubly refracting prism described in (117), two images are seen at once in different positions, every part of one of which is comple-

* The crystal is right-handed or left handed according as the Ordinary or the Extraordinary ray is of the first of these kinds. Sometimes (as in macled quartz, or amethyst) the two species of quartz are mixed: the optical phenomena which the mixture presents are very remarkable. It is to be observed (as a consequence of what is stated in the text) that in the direction of the axis the two rays, circularly polarized in opposite ways, are transmitted with different velocities: no mechanical theory has yet been formed which will completely account for this. (See however Mr Tovey's papers in the *Philosophical Magazine.*) It is remarkable that several fluids (as turpentine, sugar and water, &c.) possess this property, and even the vapour of turpentine: and apparently in all directions.

mentary to the corresponding part of the other. For, one pencil emerging from the prism consists only of vibrations perpendicular to the plane of refraction of the prism, and therefore presents to the eye the same image as the analyzing plate in a given position: the other consists only of vibrations in the plane of refraction, and therefore presents the same image as the analyzing plate in the position differing 90^0 from the former.

PROP. 39. Glass under pressure possesses double refraction.

176. This was experimentally shewn by M. Fresnel in the following manner. A number of prisms were placed as in fig. 38, and to prevent loss of light a fluid of nearly the same refractive power was dropped between the adjacent surfaces. The ends of *A, B, C, D,* were then violently pressed by means of screws. On passing a ray of light through the combination it was divided into two, one polarized in the plane of the paper and the other in the perpendicular plane.

177. It is found also that pressure affects the separation of the two rays in crystals which possess the property of double refraction. This leads to the presumption that double refraction is produced generally by a state of mechanical constraint in the particles of bodies.

178. According to our preceding theories, since compressed glass possesses double refraction, it ought, when properly interposed between the polarizing and analyzing plate, to exhibit colours. This may be seen on squeezing by means of a screw a piece of glass and holding it in the apparatus. But it may be best exhibited by taking a thick piece of plate glass which is polished at the edges, and bending it by a weight or a screw pressing the middle, and in that state placing it edgeways between the polarizing and analyzing plates at an angle of 45^0 to each plane of reflection. A black line is seen along the middle extending the whole length, with stripes more and more coloured on each side: the number of stripes is greatest in the middle of the length (perhaps six dark and as many bright): towards the ends the

stripes become broader and fewer, and the ends are wholly black. It is plain here that the central black line is seen in those parts which suffer no strain; and that those which are extended as well as those which are compressed possess double refraction. On putting a plate of mica across it, with the plane of its axes, either parallel or perpendicular to the plane of the glass, and comparing its effects on the fringes with its effects on the rings of Iceland spar, &c., it is found that the double refraction of the compressed parts is of the same kind as that of a negative crystal, and that of the extended parts of the same kind as that of a positive crystal, the axis being supposed to lie in the direction of the length of the plate glass.

179. It is found also that if glass is heated in one part, or if it is heated generally and cooled in one part, or if it is made nearly red hot and suddenly cooled by placing between cold irons, &c., it possesses the property of exhibiting beautiful colours divided by black brushes, &c., when placed between the polarizing and analyzing plates. There is no doubt that the glass is here in a state of mechanical constraint. On turning the glass, the black brushes are seen to pass by turns over every part. This determines the plane of polarization of the rays at every part of the glass, since at that point (172) ϕ must $= 0, 90^0$, &c.: that is, the two planes of polarization are then parallel and perpendicular to the plane of first polarization.

180. Between constrained glass and crystals there is however one important difference. The rings, &c. exhibited by crystals respect a direction which is independent of the size of the specimen or the part from which it is taken. The smallest fragment of crystal properly shaped may be made to exhibit the rings as well as the largest. In constrained glass, on the contrary, the rings and brushes cannot be seen, except the specimen is placed so far from the eye that the whole can be seen at once: and then we perceive an effect, not similar to that produced by rays passing in different directions through the same crystal, but to that produced by a number of crystals of different doubly refractive power arranged in different positions and then united into one system.

181. It may be useful to examine the effects of the three agents which are necessary, and in a particular order, for the exhibition of these appearances: namely the polarizing plate, the analyzing plate, and the interposed crystal. We shall begin with the last.

It is necessary that there should be an interposed body for the purpose of altering the nature of the ray in order to make it reflexible at the analyzing plate. We know of only two ways in which this can be done. One is, by resolving the vibrations into two parts and suppressing one of them. This is done if a plate of tourmaline with its axis inclined 45° to both planes of reflection is interposed; light is then reflected from the analyzing plate. The other is, by resolving the vibrations into two parts and retarding one; the retardation being either constant (measured by its effect on the phases), or varying with the colour of the light and with the direction of the ray. The first of these is done when Fresnel's rhomb is interposed: the second when a crystal or any body possessing double refraction is interposed. In either of these cases, the nature of the light compounded of these two parts as they emerge is different from that of the light which enters. But, in the former case, the change in the nature of the emergent light is sensibly the same for all rays nearly in one direction; in the latter case, it is very different even for rays whose inclinations did very little. When the planes of reflection at the polarizing and analyzing plate coincide, the same contrivance is necessary in order to make the light less capable or wholly incapable of reflection at the analyzing plate.

It is necessary for the exhibition of coloured rings that there should be an analyzing plate; because, in all the cases of resolution that we know, the intensity of the light produced by the union of the resolved parts, after one of them has been retarded, is constant (155), and equal to that of the light before it was resolved. But by again resolving this united light according to a certain law into two parts, and preserving one, it is probable that in different directions the preserved part will have different values, inasmuch as the nature of the light emerging from the crystal in different

directions is different. And this in fact is the origin of the coloured curves. But there is no necessity that the resolution should be (as we have commonly supposed) into two sets of vibrations at right angles to each other, of which only one is preserved. For instance, if Fresnel's rhomb is interposed between the crystal and the analyzing plate, coloured rings* of a different kind are seen. This artifice amounts to the same as resolving the light when it emerges from the crystal into two rays, both elliptically polarized, of the opposite kinds mentioned in (141), and preserving only one. If any other kind of resolution could be conceived, it would serve as well for exhibiting coloured rings, of forms probably different.

With respect to the necessity for a polarizing plate, there is a little more difficulty. We see from theory, as well as from observation, that light consisting of vibrations parallel to a certain plane will, after passing through the crystal and undergoing analyzation, exhibit coloured rings. But we see also that other kinds of light will do as well. For instance, circularly or elliptically polarized light (of which one has lost all trace of polarization according to the usual tests, and the other has but imperfect traces) will exhibit rings. Common light however will not exhibit rings.

182. It becomes then a matter of interest to inquire what is the difference between common light, and the class comprehending plane-polarized, circularly polarized, and elliptically polarized light. Now for a given colour of light, (that

* There is no difficulty in investigating their form. Each of the rays emerging from the crystal is to be resolved into two sets of vibrations, one parallel and one perpendicular to the plane of reflection in the rhomb: the phase of the former is to be accelerated 90^0; and the light emerging from the rhomb is to be resolved as usual at the analyzing plate. If the incident light is circularly polarized, and a uniaxal crystal is interposed, the rings are circular without any brush or cross: the center is bright or black according as the two Fresnel's rhombs have their planes of reflection coincident or at right angles to each other. If a biaxal crystal is interposed, the rings are uninterrupted, as there is no brush of any kind. This experiment is worthy of notice, as being the only one (so far as we know) in which the rings, and especially the lemniscates, are seen in their whole extent without interruption. Instead of placing a Fresnel's rhomb between the crystal and the analyzing plate, it is more convenient to use a plate of mica which retards one ray more than the other by $\frac{\lambda}{4}$: and to place it with the plane of its axes inclined 45^0 to the plane of polarization.

is, where the length of waves is invariable), and for a given intensity of light (that is, where the coefficient of vibration is invariable), the most general kind of light that we can conceive is elliptically polarized light; inasmuch as the union of any number of vibrations in any directions and following each other at any intervals will produce elliptically polarized light. Common light therefore must be elliptically polarized (including in this term plane and circularly polarized). The phænomena of interference, which are exhibited in every respect as well with common light as with polarized light, require us to allow that many successive vibrations are exactly similar to each other. For instance, on examining Newton's rings, when the incident light is that of a spirit-lamp, or that from any one point of the solar spectrum (which are nearly homogeneous), fifty or sixty rings may be seen very well, and perhaps more in favourable circumstances. Since the sixtieth of these rings is produced by the interference of a wave with the sixtieth following wave, we must conclude not only that sixty successive waves are exactly similar, but that a large multiple of sixty successive waves are exactly similar. The state of the investigation then at present is this. The phænomena of interference, combined with our most general ideas on the nature of light, compel us to suppose that common light consists of elliptic vibrations, many of which in succession are exactly similar. The difference between the phænomena of polarization (with a crystal and an analyzing plate) exhibited by common light and by elliptically polarized light, shews that common light* does not consist of an indefinite succession of similar elliptic vibrations.

183. The only supposition that seems able to reconcile these conclusions is this. *Common light consists of successive*

* We have not mentioned here the law discovered by the French philosophers, that if two streams of common light from the same source were polarized in planes perpendicular to each other, and afterwards brought to the same plane of polarization, they would not interfere; but if two streams of polarized light from the same source were treated in the same way, they would interfere. The fact is, that the observing of rings, &c. in crystals is far the best way of making the experiment: the crystal which has double refraction exhibits the two rays polarized in perpendicular planes and the analyzing plate brings them to the same plane of polarization.

series of elliptical vibrations (including in this term plane and circular vibrations), all the vibrations of each series being similar to each other, but the vibrations of one series having no relation to those of another. The number of vibrations in each series must amount to at least several hundreds; but the series must be so short that several hundred series enter the eye in every second of time.

It must be observed that a gradual change in the nature of the vibrations is inadmissible. If, for instance, we supposed the vibrations elliptical, and supposed the ellipse to revolve uniformly about its center, it would be found that the vibrations in each plane could be resolved into two whose lengths of wave were different; and compounding the corresponding vibrations in perpendicular planes, we should have two rays of elliptically polarized light of different colours.

As a simple instance of our general supposition, suppose 1000 similar vibrations in one plane to be followed by 1000 vibrations, of magnitudes equal to the former, in the plane at right angles to the former plane; then 1000 in the same plane as at first, &c. The succession of similar waves would be sufficient to give all the phænomena of interferences in perfection. At the same time, no colours would be exhibited with a crystal and an analyzing plate. For the first series alone would give rings and colours, but the second would give rings, &c. with intensities exactly complementary* to the former: and as these would enter the eye in such rapid succession that we could not distinguish them, we should only perceive the combined effect, which would be a uniform white.

184. We have always spoken of the *colour* of a ray as if it alone were sufficient to identify the nature of the ray. Perhaps, however, it would have been better to consider a ray as defined by its *refrangibility*. The remarkable experiment of Fraunhöfer (84), from which it appears that the interruptions in the spectrum formed by interference corre-

* This is seen in our expressions (155) by putting $\phi + 90^0$ instead of ϕ, and $a - 90^0$ instead of a.

spond exactly to those in the spectrum formed by refraction, seems decisive as to this point, that rays of the same refrangibility are produced by waves of the same length. It has, however, long been the opinion of some philosophers that there are rays of different colours which have the same degree of refrangibility, and that there are rays of the same colour with different degrees of refrangibility. Taking this as established, the conclusion seems to be, that colour does not depend on the length of a wave, but probably on some other circumstance, as perhaps the nature of the vibration. The law of vibration may be, not that of a cycloidal pendulum (as we have all along supposed), but something slightly different. It may be that the effect of an absorbing medium is to suppress all that part of the vibration which follows that law, and to allow only the other to pass. These, however, are very vague conjectures, which can scarcely be examined till our knowledge of the subject in question is much more extensive than it is at present.

TABLE of the NUMERICAL VALUES of the INTEGRALS.

$$\int_s \cos\frac{\pi}{2}s^2 \quad \text{and} \quad \int_s \sin\frac{\pi}{2}s^2$$

[referred to in Article 74].

Limits of integration.	$\int_s \cos\frac{\pi}{2}s^2$	$\int_s \sin\frac{\pi}{2}s^2$	Limits of integration.	$\int_s \cos\frac{\pi}{2}s^2$	$\int_s \sin\frac{\pi}{2}s^2$
From $s=0$			From s 0		
to $s=0,1$	0,0999	0,0006	to s 2,9	0,5627	0,4098
0,2	0,1999	0,0042	3,0	0,6061	0,4959
0,3	0,2993	0,0140	3,1	0,5621	0,5815
0,4	0,3974	0,0332	3,2	0,4668	0,5931
0,5	0,4923	0,0644	3,3	0,4061	0,5191
0,6	0,5811	0,1101	3,4	0,4388	0,4294
0,7	0,6597	0,1716	3,5	0,5328	0,4149
0,8	0,7230	0,2487	3,6	0,5883	0,4919
0,9	0,7651	0,3391	3,7	0,5424	0,5716
1,0	0,7803	0,4376	3,8	0,4485	0,5654
1,1	0,7643	0,5359	3,9	0,4226	0,4750
1,2	0,7161	0,6229	4,0	0,4986	0,4202
1,3	0,6393	0,6859	4,1	0,5739	0,4754
1,4	0,5439	0,7132	4,2	0,5120	0,5628
1,5	0,4461	0,6973	4,3	0,4497	0,5537
1,6	0,3662	0,6388	4,4	0,4385	0,4620
1,7	0,3245	0,5492	4,5	0,5261	0,4339
1,8	0,3342	0,4509	4,6	0,5674	0,5158
1,9	0,3949	0,3732	4,7	0,4917	0,5668
2,0	0,4886	0,3432	4,8	0,4340	0,4965
2,1	0,5819	0,3739	4,9	0,5003	0,4347
2,2	0,6367	0,4553	5,0	0,5638	0,4987
2,3	0,6271	0,5528	5,1	0,5000	0,5620
2,4	0,5556	0,6194	5,2	0,4390	0,4966
2,5	0,4581	0,6190	5,3	0,5078	0,4401
2,6	0,3895	0,5499	5,4	0,5573	0,5136
2,7	0,3929	0,4528	5,5	0,4785	0,5533
2,8	0,4678	0,3913	to s ∞	0,5	0,5

TABLE of the NUMERICAL VALUES of the DEFINITE INTEGRAL

$$\phi(n) = \frac{4}{\pi} \int_w \sqrt{(1-w^2)} \cos nw, \text{ from } w=0 \text{ to } n=1,$$

for different values of n;

[referred to in Article 86].

n	$\phi(n)$	n	$\phi(n)$	n	$\phi(n)$
0,0	+ 1,0000	4,0	− ,0330	8,0	+ ,0587
0,2	+ ,9950	4,2	− ,0660	8,2	+ ,0629
0,4	+ ,9801	4,4	− ,0922	8,4	+ ,0645
0,6	+ ,9557	4,6	− ,1116	8,6	+ ,0634
0,8	+ ,9221	4,8	− ,1244	8,8	+ ,0600
1,0	+ ,8801	5,0	− ,1310	9,0	+ ,0545
1,2	+ ,8305	5,2	− ,1320	9,2	+ ,0473
1,4	+ ,7742	5,4	− ,1279	9,4	+ ,0387
1,6	+ ,7124	5,6	− ,1194	9,6	+ ,0291
1,8	+ ,6461	5,8	− ,1073	9,8	+ ,0190
2,0	+ ,5767	6,0	− ,0922	10,0	+ ,0087
2,2	+ ,5054	6,2	− ,0751	10,2	− ,0013
2,4	+ ,4335	6,4	− ,0568	10,4	− ,0107
2,6	+ ,3622	6,6	− ,0379	10,6	− ,0191
2,8	+ ,2927	6,8	− ,0192	10,8	− ,0263
3,0	+ ,2261	7,0	− ,0013	11,0	− ,0321
3,2	+ ,1633	7,2	+ ,0151	11,2	− ,0364
3,4	+ ,1054	7,4	+ ,0296	11,4	− ,0390
3,6	+ ,0530	7,6	+ ,0419	11,6	− ,0400
3,8	+ ,0067	7,8	+ ,0516	11,8	− ,0394
4,0	− ,0330	8,0	+ ,0587	12,0	− ,0372

CAMBRIDGE: PRINTED AT THE UNIVERSITY PRESS.